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Characterizations of Matrix Ring

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Abstract: *This paper presents a nice characterization of matrix ring, especially on row and column finite matrix ring. The characterization is "A ring R satisfies the ascending chain condition on right ideals if and only if every element in $CFM(R)$ is conjugate to an element in $RCFM(R)$". Then I have tried to focus that if each element is invertible, the above characterization is exists or not.*

Keywords: $CFM(R)$, $RCFM(R)$, *conjugating matrix, invertible matrix.*

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I. Introduction

Here, we have tried to combine the concepts of linear algebra and abstract algebra. In linear algebra, any linear transformation can be formulated by matrix and in abstract algebra, if a ring R is associative with identity, then an R-module homomorphism which is related with the free right R-module is also represented by matrix. Let F_R be the free, right $R-$module on K generators, and consider the elements F_R as column vectors with K co-ordinates but with only finitely many of the co-ordinates non-zero. Then $End(F_R) \cong CFM(R)$ where each column has only finitely many non-zero entries and $CFM_K(R)$ is called the ring of column finite matrices. Let $RCFM_K(R)$ is the sub ring of $CFM_K(R)$ consisting of matrices which are both row and column finite. Recently many authors have studied about $RCFM(R)$ and $CFM(R)$. At first we discus "every element of $CFM(R)$ is conjugate to an element in $RCFM(R)$, when R is a ring with A C C ". Then try to elucidate another characterization "An invertible matrix in $CFM(R)$ that conjugates the matrix into $RCFM(R)$ when the ring R has ACC on right ideals". After that we show that " $RCFM(R)$ is clean if and only if $CFM(R)$ is clean when R is a ring with ACC on right ideals".

1.1 $CFM_k(R)$ **:** The ring of column finite matrices $CFM_k(R)$ whose entries are indexed by $k \times k$, and whose columns each contain only finitely many non-zero entries.

1.2 $RFM_k(R)$**:** The ring of row finite matrices $RFM_k(R)$ whose entries are indexed by $k \times k$, and whose rows each contain only finitely many non-zero entries.

1.3 $RCFM_k(R)$**:** The intersection of the row finite matrix rings and column finite matrix rings also forms a ring, which is denoted by $RCFM_k(R)$.

II. Conjugation

In this section we have worked out the first characterization. Let $F_R = R^{(Z_+)}$. If $T \in End(F_R)$ and B is a basis of F_R, then $T_B = CFM_{z_+}(R)$ where z_+ is a positive integer and $Supp_B(x)$ is the support of a vector $x \in F_R$ when written in the basis B. To prove the characterizations some important definitions are given below:

2.1 Conjugation of a matrix: In a commutative ring R two matrices $M, N \in M_n(R)$ are called conjugate, when there is a matrix S, such that $M = S^{-1}NS$. In linear algebra, a transformation $N \mapsto S^{-1}NS$ is called similarity transformation or **conjugation of the matrix** N.

2.2 Theorem: If R is a ring, then R satisfies the ACC (ascending chain condition) on right ideals if and only if each matrix in $CFM(R)$ and $RCFM(R)$ is conjugacy.

Proof: By the contradiction method. We try to prove if part. First we let a ring R does not satisfy the ascending chain condition on right ideal, Therefore, we have,

$$I_1 \not\subseteq I_2 \not\subseteq \cdots \text{of } R.$$

Now let $a_i \in I_i / I_{i-1}$; $i \in Z_+$. Then we can write,

$$a_1R \not\subseteq a_1R + a_2R \not\subseteq a_1R + a_2R + a_3R \not\subseteq \cdots \text{of right ideals.}$$

Now we combine the concepts of abstract algebra and linear algebra.
Let $c = \{v_1, v_2, \cdots\}$ be a fixed basis for F_R.
Let T be the transformation defined by

$T(v_i) = v_1 a_i$, for each $i \in Z_+$.

It is clear that T_c is not row finite where T_c is the matrix with the a_i along the top row, and zeros elsewhere.
In linear algebra, we know that

$$im(T) = v_1(a_1R + a_2R + \cdots).$$

By contradiction, suppose $B = \{w_1, w_2, \cdots\}$ is a basis for F_R under which T_B is a row and column finite matrix.
We can write v_1 as a finite linear combination in the w_j, say

$$v_1 = \sum_{j=1}^{n} w_j r_j \text{ ; for some } n \in Z_+, r_j \in R, j \le n.$$

In particular, $im(T) \subseteq v_1R \subseteq \bigoplus_{j=1}^{n} w_jR$.

Since T_B is row finite, the elements $\{w_1, \cdots, w_n\}$ can only appear to $\sup p_B(T(w_j))$, for $j \in Z_+$. Suppose, $T(w_j) = 0$ for $j > n$ (making n large enough).
There is some $m \in Z_+$ so that $B = \{w_1, \cdots, w_n\}$ can be written as linear combinations in the set $\{v_1, \cdots, v_m\}$.
In particular, for $j \le n$ we have

$$T(w_j) \in T(v_1)R + \cdots + T(v_m)R = v_1(a_1R + \cdots + a_mR), \text{ and hence}$$

$$im(T) \subseteq v_1(a_1R + \cdots + a_mR) \not\subseteq v_1(a_1R + a_2R + \cdots), \text{ which is a}$$

contradiction.
Therefore, T cannot be written as a row and column finite matrix.

Now, we prove only if part. Let T be an endomorphism of F_R. Given any basis B of F_R, we write $T_{B,(i,j)}$ for the $(i, j)-$ entry of T_B. Let $B_1 = \{v_{1,1}, v_{2,1}, v_{3,1}, \cdots\}$ be a fixed basis for F_R. For each $k \in Z_+$ we will construct a basis $B_k = \{v_{1,k}, v_{2,k}, \cdots\}$ satisfying the following five conditions:

1. The first k vectors of both B_k and B_{k+1} agree.
2. For any $k \in Z_+$, the span of the first n vectors from B_k equals the span of the first n vectors of B_{k+1}.
3. The first $k-1$ rows of both T_{B_k} and $T_{B_{k+1}}$ agree.

4. When passing from T_{B_k} to $T_{B_{k+1}}$, the first k columns do not increase in length.
5. The first k rows of $T_{B_{k+1}}$ are finite.

Before we prove the existence of such bases, let us look at some consequences of these conditions. First, let $B_\Delta = \{v_{1,1}, v_{2,2}, v_{3,3}, \cdots\}$ be the set of diagonal elements arising from these bases. Then condition (1) implies that for any $k \in Z_+$ we also have $B_\Delta = \{v_{1,k}, v_{2,k}, \cdots, v_{k,k}, v_{k+1,k+1}, v_{k+2,k+2}, \cdots\}$. So, in some sense, $B_\Delta = \lim_{n\to\alpha} B_n$. Second, by condition (2) we see that B_Δ is actually a basis for F_R.

Next, assume that the k th column of T_{B_k} has length no longer than $d \geq k$. Then condition (4) implies that the k th column of each of the matrices $T_{B_k}, T_{B_{k+1}}, \cdots$ is of length $\leq d$. We then see that the k th column of each of the matrices $T_{B_{d+1}}, T_{B_{d+2}}, \cdots$ agree, since by condition (3) the first d rows agree. Also, since the first d vectors of B_d and B_Δ agree, we see that this column is also the k th column of T_{B_Δ}. So T_{B_Δ} is the matrix which is the limit of the matrices $T_{B_1}, T_{B_2}, \cdots$. Finally, by conditions (3) and (5) we see that the rows of T_{B_Δ} must be finite, and hence B_Δ is the needed basis.

Now we prove existence. Working by induction, we may suppose that $B_k = \{v_{1,k}, v_{2,k}, v_{3,k}, \cdots\}$ has been constructed for some $k \geq 1$. By hypothesis, the first $k-1$ rows of T_{B_k} are finite, so we set $p_k = \max\ \{ j | T_{B_k,(i,j)} \neq 0$ for some $i \in [1, k-1]\}$. In other words p_k, is the maximum length of the first $k-1$ rows. (If $k=1$ we set $p_k = 1$.) Set $m_k = \max\{k+1, p_k\}$, and let J_k be the right ideal generated by the entries $T_{B_k,(k,j)}$ for $j \geq m_k + 1$. Since R satisfies ascending chain condition on right ideals, there is some integer $n_k \geq m_k + 1$ such that J_k is generated by $T_{B_k,(k,j)}$ for $j \in [m_k + 1, n_k]$. Finally, define $B_{k+1} = \{v_{1,k+1}, v_{2,k+1}, \cdots\}$. For $l \leq n_k$, put $v_{l,k+1} := v_{l,k}$. Put $l > n_k$, put $v_{l,k+1} := v_{l,k} + \sum_{i=m_k+1}^{n_k} v_{i,k} c_{i,l}$ for some $c_{i,l} \in R$. In other words, the change of basis matrix is of the form $U_k := \begin{pmatrix} I_{m_k} & 0 & 0 \\ 0 & I_{n_k - m_k} & C \\ 0 & 0 & I_{Z_+} \end{pmatrix}$, where C is the matrix formed from the constants $c_{i,l}$ and $I_\bullet$ is the $\bullet \times \bullet$ identity matrix. And we get $U^{-1}{}_k := \begin{pmatrix} I_{m_k+1} & 0 & 0 \\ 0 & I_{n_k - m_k} & -C \\ 0 & 0 & I_{Z_+} \end{pmatrix}$. Because $T_{B_{k+1}} = U^{-1}{}_k T_{B_k} U_k$, after these matrix multiplication now prove that our five conditions are met.

Right multiplication by U_k corresponds to column operations. So $T_{B_k} U_k$ is the matrix formed by taking T_{B_k} and adding $c_{i,l}$ times the i th column to the l th column (for $i \in [m_k + 1, n_k]$ and $l > n_k$). But because J_k is generated by the entries along the k th row in these columns, choose the $c_{i,l}$ so that the k th row of $T_{B_k} U_k$ becomes 0 after the n_k th column. (In other words, we “column reduce” along the k th row, after a specified point.) Also notice that the first $k-1$ rows of T_{B_k} are not affected by right multiplication by U_k, because we chose $m_k \geq p_k$. Therefore, the first k rows of $T_{B_k} U_k$ are finite. Also, because $m_k > k$, the first k rows of $T_{B_k} U_k$ are unchanged by left multiplication by U_k^{-1}, and hence condition (5) holds. These facts then imply that the first $k-1$ rows of T_{B_k} and $T_{B_{k+1}}$ agree, yielding condition (3). Conditions (1) and (2) are obvious from the construction. Finally, right multiplication by U_k doesn't change the first k columns, and left multiplication by U_k^{-1} corresponds to adding rows upwards (since this matrix is upper-triangular) and so cannot increase column length. Hence condition (4) also holds.

III. Invertible

3.1 Invertible matrix: A $n \times n$ square matrix A is called invertible if there exists an $n \times n$ square matrix B such that $AB = BA = I_n$, where I_n denotes the $n \times n$ identity matrix.

3.2 Proposition: Let M be a countably generated module over a ring where the ring has ascending chain conditions on right ideals. Let $\cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$ is a free resolution of M with each F_n countably generated. (Such a resolution is possible from our work above.) Then there exists bases B_n of F_n, for each $n \in \mathrm{N}$, so that the maps $\varphi_{n+1} : F_{n+1} \to F_n$ are represented by row and column finite matrices in these bases. In other words, each element $b \in B_n$ occurs in the support of only finitely many elements of $\varphi_{n+1}(B_{n+1})$.

Now it is tried to show that the invertibility $CFM(R)$ of $RCFM(R)$ and are conjugate.

Let $T_1, \ldots, T_r \in CFM(R)$, for some $r \in Z_+$.

3.3 Theorem: Let R be a ring having ACC on right ideals. The invertible of a set of elements in $CFM(R)$ that conjugates the set into $RCFM(R)$.

Proof: First we want to inductively construct a sequence of bases for F_R so that the diagonal set is a basis with the right properties. We have that since there are only countably many elements $T_1, T_2, \cdots \in End(F_R) \cong CFM(R)$, there are also only countably many rows that we need to column reduce. By a diagonalization well-ordering, we order all the rows $\{r_{m,n,B}\}$: $r_{m,n,B} \le r_{m',n',B}$ if and only if $m+n < m'+n'$, or $m+n = m'+n'$ and $m \le m'$, where $r_{m,n,B}$ is the mth row of $T_{n,B}$, under a basis B for F. And for well-ordering, it is restated conditions (3) through (5) of [4, Theorem2] to prove the theorem.

The conjugating matrices are defined exactly as before, and the constants are chosen so that we column reduce the row $r_{m',n',B}$. The rest of the proof is unchanged.

3.4 Clean ring: A ring is said to be clean if every element in the ring can be written as the sum of a unit and an idempotent of the ring.

3.5Theorem: Let R be a ring with ACC on right ideals, then $CFM(R)$ is clean if and only if $RCFM(R)$ is clean.

Proof: Let R be a ring with ACC on right ideals. As $CFM(R)$ is clean, for any element $x \in CFM(R)$ there exists $e, u, v \in CFM(R)$ so that

$$x = u + e, e^2 = e \qquad \text{……………….(1)}$$

$$\text{and } uv = vu = 1. \qquad \text{……………..(2)}$$

By conjugating these equations by some element $\sigma \in U(CFM(R))$ so that the equations (1) and (2) becomes

$$x^\sigma = u^\sigma + e^\sigma, (e^2)^\sigma = e^\sigma$$

$$\text{and } u^\sigma v^\sigma = v^\sigma u^\sigma = 1^\sigma = 1 \text{ hold in } RCFM(R).$$

Thus x^{σ} is clean, even in $RCFM(R)$.
If $x \in RCFM(R) \not\leq CFM(R)$,
choose σ so that $x^{\sigma} = x$ and this would show that $RCFM(R)$ is clean.

Conversely,
let R be a ring with ACC on right ideals for which $RCFM(R)$ is clean.
Given $x \in CFM(R)$ we can find $\sigma \in U(CFM(R))$ so that $x^{\sigma} \in RCFM(R)$.
Then we can find $u \in U(RCFM(R))$ and $e^2 = e \in RCFM(R)$ so that $x^{\sigma} = u + e$. Then $x = u^{\sigma^{-1}} + e^{\sigma^{-1}}$, and hence $CFM(R)$ is clean.

IV. Conclusion

Theorem (2.2) states that a ring R which satisfies the ACC on right ideals if and only if each matrix in column finite matrix over R that conjugate to a matrix in row and column finite matrix over R. Theorem (3.3) establishes that for a ring R which satisfies the ACC on right ideals, each countable set of elements of $CFM(R)$ there exists an invertible matrix in $CFM(R)$ that conjugates the set into $RCFM(R)$. Then it follows from theorem (3.5) that a ring R with ACC on right ideals, then $CFM(R)$ is clean if and only if $RCFM(R)$ is clean.

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