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Three Unextendible Maximally Entangled Bases In $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$

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***Abstract**: The construction of an UMEB in $\mathbb{C}^{qd}\otimes\mathbb{C}^{qd}$ from an UMEB in $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$ is discussedthis paper. Three kinds of 132-member UMEBs in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$ are established from the UMEBs of $\mathbb{C}^{3}\otimes\mathbb{C}^{3}$, $\mathbb{C}^{4}\otimes\mathbb{C}^{4}$ and $\mathbb{C}^{6}\otimes\mathbb{C}^{6}$ respectively.*

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I. Introduction

In2009, Sergei Bravyi and John A.Smolin[1] generalized the notion of the UPB to the Unextendible maximally entangled bases, they proved that UMEBs do not exist in $\mathbb{C}^{2}\otimes\mathbb{C}^{2}$,and theypresented a 6-member UMEB in $\mathbb{C}^{3}\otimes\mathbb{C}^{3}$ and a 12-member UMEB in $\mathbb{C}^{4}\otimes\mathbb{C}^{4}$.

Since then,more and more people start to study UMEBs.in Chen and Shao-Ming Fei[2] gave d^2-member UMEB in $\mathbb{C}^{d}\otimes\mathbb{C}^{d'}$ $(\frac{d'}{2}<d<d')$; Hua Nan,Yuan-Hong Tao,Lin-Song Li and Jun Zhang[3] showed that there are qd^2 -member UMEBs in $\mathbb{C}^{d}\otimes\mathbb{C}^{d'}$ $(d'=qd+r, q,r\in\mathbb{Z}^{+}, r<d)$ .Mao-Sheng Li,Yan-Ling Wang and Zhu-Jun Zheng[4] concluded that there always exist a UMEB in $\mathbb{C}^{d}\otimes\mathbb{C}^{d'}$ $(d\neq d')$ .Yan-Ling Wang, Mao-Sheng Liand Shao-Ming Fei[5] have presented the construction of an UMEB in $\mathbb{C}^{qd}\otimes\mathbb{C}^{qd}$ from an UMEB in $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$,they showed that for an given N-number UMEB in $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$,there is a $\tilde{N}$ -number UMEBS in $\mathbb{C}^{qd}\otimes\mathbb{C}^{qd}$ where $\tilde{N}=q^2d^2-qd^2+qN$, but they did not show the difference among different UMEBS in $\mathbb{C}^{qd}\otimes\mathbb{C}^{qd}$.

In this paper,we show that there are three kinds of 132-member UMEBs in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$,which are constructed from 6-member UMEB in $\mathbb{C}^{3}\otimes\mathbb{C}^{3}$, 12-member UMEB in $\mathbb{C}^{4}\otimes\mathbb{C}^{4}$ and 30-member UMEB in $\mathbb{C}^{6}\otimes\mathbb{C}^{6}$ separately.

II. UMEB IN $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$

A set of states $\{|\psi_a\rangle\in\mathbb{C}^{d}\otimes\mathbb{C}^{d}, a=1\cdots n, n<d^2\}$ is called an UMEB[1] in $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$ if and only if:

(1) All states $|\psi_a\rangle$ are maximally entangled;

(2) $\langle\psi_a|\psi_b\rangle=\delta_{a,b},\ a,b=1,2,\cdots,n;$

(3) If $\langle\psi_a|\psi\rangle=0$ for all $a=1,2,\cdots,n$, then $|\psi\rangle$ cannot be maximally entangled.

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Let $\{|1\rangle,|2\rangle,\cdots,|d\rangle\}$ be the computational basis in $\mathbb{C}^{d}$, then any maximally entangled states $|\psi_a\rangle$ can be expressed as

$$|\psi_a\rangle=(I\otimes U_a)\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle\otimes|i\rangle,$$

WhereI is the $d\times d$ identity matrix, U_a is any $d\times d$ unitary matrix .A set of unitary matrices $\{U_a\in M_d(\mathbb{C})\,|\,a=1,2,\cdots,n\}$ gives an n-member UMEB in $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$ [1] ifand only if

(1) $n<d^2$

(2) $Tr(U_a^{\dagger}U_b)=d\delta_{ab}, a,b=1,2,\cdots,n$;

(3) If $Tr(U_a^{\dagger}U)=0$, for any $U\in M_d(\mathbb{C})$, $a=1,2,\cdots,n$, then U cannot be unitary.

III. UMEBS IN $\mathbb{C}^{qd}\otimes\mathbb{C}^{qd}\ (qd>2)$

Yan-Ling Wang, Mao-Sheng Liand Shao-Ming Fei[5],have established the construction of an $\tilde{N}-member$ UMEB in $\mathbb{C}^{qd}\otimes\mathbb{C}^{qd}$ from an $N-member$ UMEB in $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$, where $\tilde{N}=q^2d^2-qd^2+qN, q\in\mathbb{N}$. The detailed process is as follows:
Denote

$$U_{nm}=\sum_{k=0}^{d-1}e^{\frac{2\pi\sqrt{-1}}{d}kn}|k\oplus m\rangle\langle k|, m,n=0,1,\cdots,d-1,$$

and

$$S=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\1&0&0&\cdots&0\end{pmatrix}, W=\begin{pmatrix}1&1&1&\cdots&1\\1&\zeta_q&\zeta_q^2&\cdots&\zeta_q^{q-1}\\1&\zeta_q^2&\zeta_q^4&\cdots&\zeta_q^{2(q-1)}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&\zeta_q^{q-1}&\zeta_q^{2(q-1)}&\cdots&\zeta_q^{(q-1)^2}\end{pmatrix}$$

where $\zeta_q=e^{\frac{2\pi\sqrt{-1}}{q}}$.

Let $\{U_n, n=1,2,\cdots,N<d^2\}$ be the set of unitary matrices that give rise to the UMEB in $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$.Set $U_{nm}^{ij}=(W^iS^j)\otimes U_{nm}$, where $i=0,\cdots,q-1; j=1,\cdots,q-1; m,n=0,\cdots,d-1;$ and

$$U_n^i=W^i\otimes U_n,\ i=0,1,\cdots,q-1, n=1,2,\cdots,N<d^2.$$

then $\{U_{nm}^{ij},U_n^i\}$ give a $\tilde{N}-member$ UMEB in $\mathbb{C}^{qd}\otimes\mathbb{C}^{qd}$.

Let $\tilde{N}$ denote the number of all the above matrices in $\{U_{nm}^{ij}\}$ and $\{U_n^i\}$, then

$\tilde{N}=q(q-1)d^2+qN=q^2d^2-qd^2+qN<qd^2$.

IV. UMEBS in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$

Yan-Ling Wang, Mao-Sheng Liand Shao-Ming Fei[5] have presented a 30 member UMEB in $\mathbb{C}^{6}\otimes\mathbb{C}^{6}$ from that in $\mathbb{C}^{3}\otimes\mathbb{C}^{3}$.In this chapter, we will establish three kinds of 132-member UMEB in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$.Note that $12=3\times4$ and $12=2\times6$, and UMEBS do not exist in $\mathbb{C}^{2}\otimes\mathbb{C}^{2}$ [1], we can construct UMEBs in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$ from the UMEBs of $\mathbb{C}^{3}\otimes\mathbb{C}^{3}$, $\mathbb{C}^{4}\otimes\mathbb{C}^{4}$ and $\mathbb{C}^{6}\otimes\mathbb{C}^{6}$ separately. Each way shows 132-member UMEBs in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$.

4.1. UMEB in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$ from that in $\mathbb{C}^{3}\otimes\mathbb{C}^{3}$

Let $\{|0\rangle,|1\rangle,\cdots,|11\rangle\}$ be the computational basis in $\mathbb{C}^{12}$.We first present thecorresponding matrices $\{U_{nm}^{ij}, i=0,1,2,3, j=1,2,3, m,n=0,1,2\}$ to give 108 maximally entangled states in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$ as follows:

Now $\zeta = e^{\frac{2\pi\sqrt{-1}}{4}}$, $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$, $U_{nm} = \sum_{k=0}^{2} e^{\frac{2\pi\sqrt{-1}}{3}kn} |k\oplus m\rangle\langle k|, m,n=0,1,2$.

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^2 & 1 & \zeta^2 \\ 1 & \zeta^3 & \zeta^2 & \zeta \end{pmatrix}, S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

Denote S^j the j power of S, $j=1,2,3$ and $W^k = diag(w_{k+1,1}, w_{k+1,2}, w_{k+1,3}, w_{k+1,4}), k=0,1,2,3$, we have

$W^0 = diag(1,1,1,1),\ W^1 = diag(1,\zeta,\zeta^2,\zeta^3),\ W^2 = diag(1,\zeta^2,1,\zeta^2),\ W^3 = diag(1,\zeta^3,\zeta^2,\zeta)$ .Then we have 108 orthonormal maximally entangled states in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$ as follows:

Define

$$|\phi_{nm}^{ij}\rangle = \frac{1}{\sqrt{12}}\sum_{i=0}^{11}|i'\rangle\otimes(U_{nm}^{ij}|i'\rangle) = \frac{1}{2\sqrt{3}}\sum_{i=0}^{11}|i\rangle\otimes((W^k S^j\otimes U_{nm})|i'\rangle) \quad (1)$$

Now we illustrate the above 108 states. The first 36 states are as follows:

$$\phi_{a1} = a_1|03'\rangle + a_2|14'\rangle + a_3|25'\rangle + a_4|36'\rangle + a_5|47'\rangle + a_6|58'\rangle + a_7|69'\rangle + a_8|7{,}10'\rangle + a_9|8{,}11'\rangle + a_{10}|90'\rangle + a_{11}|10{,}1'\rangle + a_{12}|11{,}2'\rangle$$

$$\phi_{a2} = a_1|04'\rangle + a_2|15'\rangle + a_3|23'\rangle + a_4|37'\rangle + a_5|48'\rangle + a_6|56'\rangle + a_7|6{,}10'\rangle + a_8|7{,}11'\rangle + a_9|89'\rangle + a_{10}|91'\rangle + a_{11}|10{,}2'\rangle + a_{12}|11{,}0'\rangle$$

$$\phi_{a3} = a_1|05'\rangle + a_2|13'\rangle + a_3|24'\rangle + a_4|38'\rangle + a_5|46'\rangle + a_6|57'\rangle + a_7|6{,}11'\rangle + a_8|79'\rangle + a_9|8{,}10'\rangle + a_{10}|92'\rangle + a_{11}|10{,}0'\rangle + a_{12}|11{,}1'\rangle$$

where the coefficients $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12})$ take the following 12 group values:

- $(1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2)$
- $(1,\omega^2,\omega,1,\omega^2,\omega,1,\omega^2,\omega,1,\omega^2,\omega)$
- $(1,1,1,1,1,1,1,1,1,1,1,1)$
- $(\zeta,\zeta\omega,\zeta\omega^2,\zeta^2,\zeta^2\omega,\zeta^2\omega^2,\zeta^3,\zeta^3\omega,\zeta^3\omega^2,1,\omega,\omega^2)$
- $(\zeta,\zeta\omega^2,\zeta\omega,\zeta^2,\zeta^2\omega^2,\zeta^2\omega,\zeta^3,\zeta^3\omega^2,\zeta^3\omega,1,\omega^2,\omega)$
- $(\zeta,\zeta,\zeta,\zeta^2,\zeta^2,\zeta^2,\zeta^3,\zeta^3,\zeta^3,1,1,1)$
- $(\zeta^2,\zeta^2\omega,\zeta^2\omega^2,1,\omega,\omega^2,\zeta^2,\zeta^2\omega,\zeta^2\omega^2,1,\omega,\omega^2)$
- $(\zeta^2,\zeta^2\omega^2,\zeta^2\omega,1,\omega^2,\omega,\zeta^2,\zeta^2\omega^2,\zeta^2\omega,1,\omega^2,\omega)$
- $(\zeta^2,\zeta^2,\zeta^2,1,1,1,\zeta^2,\zeta^2,\zeta^2,1,1,1)$
- $(\zeta^3,\zeta^3\omega,\zeta^3\omega^2,\zeta^2,\zeta^2\omega,\zeta^2\omega^2,\zeta,\zeta\omega,\zeta\omega^2,1,\omega,\omega^2)$
- $(\zeta^3,\zeta^3\omega^2,\zeta^3\omega,\zeta^2,\zeta^2\omega^2,\zeta^2\omega,\zeta,\zeta\omega^2,\zeta\omega,1,\omega^2,\omega)$
- $(\zeta^3,\zeta^3,\zeta^3,\zeta^2,\zeta^2,\zeta^2,\zeta,\zeta,\zeta,1,1,1)$

The second 36 states are as follows:

$$\phi_{b1}=b_1|06'\rangle+b_2|17'\rangle+b_3|28'\rangle+b_4|39'\rangle+b_5|4,10'\rangle+b_6|5,11'\rangle+b_7|60'\rangle+b_8|71'\rangle+b_9|82'\rangle+b_{10}|93'\rangle+b_{11}|10,4'\rangle+b_{12}|11,5'\rangle$$
$$\phi_{b2}=b_1|07'\rangle+b_2|18'\rangle+b_3|26'\rangle+b_4|3,10'\rangle+b_5|4,11'\rangle+b_6|59'\rangle+b_7|61'\rangle+b_8|72'\rangle+b_9|80'\rangle+b_{10}|94'\rangle+b_{11}|10,5'\rangle+b_{12}|11,3'\rangle$$
$$\phi_{b3}=b_1|08'\rangle+b_2|16'\rangle+b_3|27'\rangle+b_4|3,11'\rangle+b_5|49'\rangle+b_6|5,10'\rangle+b_7|62'\rangle+b_8|70'\rangle+b_9|81'\rangle+b_{10}|95'\rangle+b_{11}|10,3'\rangle+b_{12}|11,4'\rangle$$

where the coefficients $(b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10},b_{11},b_{12})$ take the following 12 group values:

- $(1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2)$
- $(1,\omega^2,\omega,1,\omega^2,\omega,1,\omega^2,\omega,1,\omega^2,\omega)$
- $(1,1,1,1,1,1,1,1,1,1,1,1)$
- $(\zeta^2,\zeta^2\omega,\zeta^2\omega^2,\zeta^3,\zeta^3\omega,\zeta^3\omega^2,1,\omega,\omega^2,\zeta,\zeta\omega,\zeta\omega^2)$
- $(\zeta^2,\zeta^2\omega^2,\zeta^2\omega,\zeta^3,\zeta^3\omega^2,\zeta^3\omega,1,\omega^2,\omega,\zeta,\zeta\omega^2,\zeta\omega)$
- $(\zeta^2,\zeta^2,\zeta^2,\zeta^3,\zeta^3,\zeta^3,1,1,1,\zeta,\zeta,\zeta)$
- $(1,\omega,\omega^2,\zeta^2,\zeta^2\omega,\zeta^2\omega^2,1,\omega,\omega^2,\zeta^2,\zeta^2\omega,\zeta^2\omega^2)$
- $(1,\omega^2,\omega,\zeta^2,\zeta^2\omega^2,\zeta^2\omega,1,\omega^2,\omega,\zeta^2,\zeta^2\omega^2,\zeta^2\omega)$
- $(1,1,1,\zeta^2,\zeta^2,\zeta^2,1,1,1,\zeta^2,\zeta^2,\zeta^2)$
- $(\zeta^2,\zeta^2\omega,\zeta^2\omega^2,\zeta,\zeta\omega,\zeta\omega^2,1,\omega,\omega^2,\zeta^3,\zeta^3\omega,\zeta^3\omega^2)$
- $(\zeta^2,\zeta^2\omega^2,\zeta^2\omega,\zeta,\zeta\omega^2,\zeta\omega,1,\omega^2,\omega,\zeta^3,\zeta^3\omega^2,\zeta^3\omega)$
- $(\zeta^2,\zeta^2,\zeta^2,\zeta,\zeta,\zeta,1,1,1,\zeta^3,\zeta^3,\zeta^3)$

The third 36 states are as follows:

$$\phi_{c1}=c_1|09'\rangle+c_2|1,10'\rangle+c_3|2,11'\rangle+c_4|30'\rangle+c_5|41'\rangle+c_6|52'\rangle+c_7|63'\rangle+c_8|74'\rangle+c_9|85'\rangle+c_{10}|96'\rangle+c_{11}|10,7'\rangle+c_{12}|11,8'\rangle$$
$$\phi_{c2}=c_1|0,10'\rangle+c_2|1,11'\rangle+c_3|29'\rangle+c_4|31'\rangle+c_5|42'\rangle+c_6|50'\rangle+c_7|64'\rangle+c_8|75'\rangle+c_9|83'\rangle+c_{10}|97'\rangle+c_{11}|10,8'\rangle+c_{12}|11,6'\rangle$$
$$\phi_{c3}=c_1|0,11'\rangle+c_2|19'\rangle+c_3|2,10'\rangle+c_4|32'\rangle+c_5|40'\rangle+c_6|51'\rangle+c_7|65'\rangle+c_8|73'\rangle+c_9|84'\rangle+c_{10}|98'\rangle+c_{11}|10,6'\rangle+c_{12}|11,7'\rangle$$

where the coefficients $(c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8,c_9,c_{10},c_{11},c_{12})$ take the following 12 group values:

- $(1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2)$
- $(1,\omega^2,\omega,1,\omega^2,\omega,1,\omega^2,\omega,1,\omega^2,\omega)$
- $(1,1,1,1,1,1,1,1,1,1,1,1)$
- $(\zeta^3,\zeta^3\omega,\zeta^3\omega^2,1,\omega,\omega^2,\zeta,\zeta\omega,\zeta\omega^2,\zeta^2,\zeta^2\omega,\zeta^2\omega^2,)$
- $(\zeta^3,\zeta^3\omega^2,\zeta^3\omega,1,\omega^2,\omega,\zeta,\zeta\omega^2,\zeta\omega,\zeta^2,\zeta^2\omega^2,\zeta^2\omega)$
- $(\zeta^3,\zeta^3,\zeta^3,1,1,1,\zeta,\zeta,\zeta,\zeta^2,\zeta^2,\zeta^2)$
- $(\zeta^2,\zeta^2\omega,\zeta^2\omega^2,1,\omega,\omega^2,\zeta^2,\zeta^2\omega,\zeta^2\omega^2,1,\omega,\omega^2)$
- $(\zeta^2,\zeta^2\omega^2,\zeta^2\omega,1,\omega^2,\omega,\zeta^2,\zeta^2\omega^2,\zeta^2\omega,1,\omega^2,\omega)$
- $(\zeta^2,\zeta^2,\zeta^2,1,1,1,\zeta^2,\zeta^2,\zeta^2,1,1,1)$
- $(\zeta,\zeta\omega,\zeta\omega^2,1,\omega,\omega^2,\zeta^3,\zeta^3\omega,\zeta^3\omega^2,\zeta^2,\zeta^2\omega,\zeta^2\omega^2)$
- $(\zeta,\zeta\omega^2,\zeta\omega,1,\omega^2,\omega,\zeta^3,\zeta^3\omega^2,\zeta^3\omega,\zeta^2,\zeta^2\omega^2,\zeta^2\omega)$
- $(\zeta,\zeta,\zeta,1,1,1,\zeta^3,\zeta^3,\zeta^3,\zeta^2,\zeta^2,\zeta^2)$

Next,we present the corresponding matrices $\{U_n^i, i=0,1,2,3, n=1,2,3,4,5,6\}$ to give 24 maximally entangled statesin $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$:

Let $\{U_n = I-(1-e^{i\theta})|\psi_n\rangle\langle\psi_n|, n=1,2,\cdots,6\}$ be the set of unitary matrices that give rise to the 6-member UMEB in $\mathbb{C}^3\otimes\mathbb{C}^3$, where

$$|\psi_{1,2}\rangle=\frac{1}{N}(|0\rangle\pm\phi|1\rangle)$$
$$|\psi_{3,4}\rangle=\frac{1}{N}(|1\rangle\pm\phi|2\rangle)$$
$$|\psi_{5,6}\rangle=\frac{1}{N}(|2\rangle\pm\phi|0\rangle)$$

with $\phi=\frac{1+\sqrt{5}}{2}, N=\sqrt{1+\phi^2}, \cos\theta=-\frac{7}{8}$.

Denote $U_n^i=W^i\otimes U_n$ through directly calculation, we have

$$U_n^i=I_{12}^i-(1-e^{i\theta})(|\psi_n\rangle\langle\psi_n|+|\psi_{n+6}\rangle\langle\psi_{n+6}|+|\psi_{n+12}\rangle\langle\psi_{n+12}|+|\psi_{n+18}\rangle\langle\psi_{n+18}|),$$

$i=0,1,2,3, n=1,2,\cdots,6$ and

$$|\psi_{1,2}\rangle=\frac{1}{N}(|0\rangle\pm\phi|1\rangle),\ |\psi_{3,4}\rangle=\frac{1}{N}(|1\rangle\pm\phi|2\rangle),\ |\psi_{5,6}\rangle=\frac{1}{N}(|2\rangle\pm\phi|0\rangle)$$
$$|\psi_{7,8}\rangle=\frac{1}{N}(|3\rangle\pm\phi|4\rangle),\ |\psi_{9,10}\rangle=\frac{1}{N}(|4\rangle\pm\phi|5\rangle),\ |\psi_{11,12}\rangle=\frac{1}{N}(|5\rangle\pm\phi|3\rangle)$$
$$|\psi_{13,14}\rangle=\frac{1}{N}(|6\rangle\pm\phi|7\rangle),\ |\psi_{15,16}\rangle=\frac{1}{N}(|7\rangle\pm\phi|8\rangle),\ |\psi_{17,18}\rangle=\frac{1}{N}(|8\rangle\pm\phi|6\rangle)$$
$$|\psi_{19,20}\rangle=\frac{1}{N}(|9\rangle\pm\phi|10\rangle),\ |\psi_{21,22}\rangle=\frac{1}{N}(|10\rangle\pm\phi|11\rangle),\ |\psi_{23,24}\rangle=\frac{1}{N}(|11\rangle\pm\phi|9\rangle)$$

$$I_{12}^0=\begin{pmatrix} I_3 & 0_3 & 0_3 & 0_3\\ 0_3 & I_3 & 0_3 & 0_3\\ 0_3 & 0_3 & I_3 & 0_3\\ 0_3 & 0_3 & 0_3 & I_3\end{pmatrix},\quad I_{12}^1=\begin{pmatrix} I_3 & 0_3 & 0_3 & 0_3\\ 0_3 & \zeta I_3 & 0_3 & 0_3\\ 0_3 & 0_3 & \zeta^2 I_3 & 0_3\\ 0_3 & 0_3 & 0_3 & \zeta^3 I_3\end{pmatrix}$$

$$I_{12}^2=\begin{pmatrix} I_3 & 0_3 & 0_3 & 0_3\\ 0_3 & \zeta^2 I_3 & 0_3 & 0_3\\ 0_3 & 0_3 & I_3 & 0_3\\ 0_3 & 0_3 & 0_3 & \zeta^2 I_3\end{pmatrix},\ I_{12}^3=\begin{pmatrix} I_3 & 0_3 & 0_3 & 0_3\\ 0_3 & \zeta^3 I_3 & 0_3 & 0_3\\ 0_3 & 0_3 & \zeta^2 I_3 & 0_3\\ 0_3 & 0_3 & 0_3 & \zeta I_3\end{pmatrix}$$

Define

$$|\psi_n^i\rangle=(I\otimes U_n^i)\frac{1}{2\sqrt{3}}\sum_{\alpha=0}^{11}|\alpha\rangle\otimes|\alpha\rangle,\ i=0,1,2,3; n=1,2,\cdots,6 \qquad (2)$$

then the above24 states $\{|\psi_n^i\rangle\}$ are all orthonormal maximally entangled states in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$. It is easy to verified that all the above 132 states in (1) and (2) construct a UMEB in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$.

4.2. UMEB in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$ from that in $\mathbb{C}^{4}\otimes\mathbb{C}^{4}$

We first present the corresponding matrices $\{U_{nm}^{ij}, i=0,1,2, j=1,2, m,n=0,1,2,3\}$ to give 96 maximally entangled statesin $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$:

Now $\zeta=e^{\frac{2\pi\sqrt{-1}}{3}}$, $\omega=e^{\frac{2\pi\sqrt{-1}}{4}}$, $U_{nm}=\sum_{k=0}^{3}e^{\frac{2\pi\sqrt{-1}}{4}kn}|k\oplus m\rangle\langle k|, m,n=0,1,2,3$ and

$$W=\begin{pmatrix}1 & 1 & 1\\ 1 & \zeta & \zeta^2\\ 1 & \zeta^2 & \zeta\end{pmatrix},\ S=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$$

Denote S^j the j power of S, , $j=1,2,3$ and $W^k=diag(w_{k+1,1},w_{k+1,2},w_{k+1,3}),k=0,1,2$,we have $W^0=diag(1,1,1),\ W^1=diag(1,\zeta,\zeta^2),\ W^2=diag(1,\zeta^2,\zeta)$. Then we have the following 96 orthonormal maximally entangled states in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$:

Define

$$|\phi_{nm}^{ij}\rangle=\frac{1}{\sqrt{12}}\sum_{i=0}^{11}|i^{'}\rangle\otimes(U_{nm}^{ij}|i^{'}\rangle)=\frac{1}{2\sqrt{3}}\sum_{i=0}^{11}|i\rangle\otimes((W^kS^j\otimes U_{nm})|i^{'}\rangle) \qquad (3)$$

Now we illustrate the above 96 states.The first 48 states are as follows:

$$\phi_{a1}=a_1|04^{'}\rangle+a_2|15^{'}\rangle+a_3|26^{'}\rangle+a_4|37^{'}\rangle+a_5|48^{'}\rangle+a_6|59^{'}\rangle+a_7|6{,}10^{'}\rangle+a_8|7{,}11^{'}\rangle+a_9|80^{'}\rangle+a_{10}|91^{'}\rangle+a_{11}|10{,}2^{'}\rangle+a_{12}|11{,}3^{'}\rangle$$

$$\phi_{a2}=a_1|05^{'}\rangle+a_2|16^{'}\rangle+a_3|27^{'}\rangle+a_4|34^{'}\rangle+a_5|49^{'}\rangle+a_6|5{,}10^{'}\rangle+a_7|6{,}11^{'}\rangle+a_8|78^{'}\rangle+a_9|81^{'}\rangle+a_{10}|92^{'}\rangle+a_{11}|10{,}3^{'}\rangle+a_{12}|11{,}0^{'}\rangle$$

$$\phi_{a3}=a_1|06^{'}\rangle+a_2|17^{'}\rangle+a_3|24^{'}\rangle+a_4|35^{'}\rangle+a_5|4{,}10^{'}\rangle+a_6|5{,}11^{'}\rangle+a_7|68^{'}\rangle+a_8|79^{'}\rangle+a_9|82^{'}\rangle+a_{10}|93^{'}\rangle+a_{11}|10{,}0^{'}\rangle+a_{12}|11{,}1^{'}\rangle$$

$$\phi_{a4}=a_1|07^{'}\rangle+a_2|14^{'}\rangle+a_3|25^{'}\rangle+a_4|36^{'}\rangle+a_5|4{,}11^{'}\rangle+a_6|58^{'}\rangle+a_7|69^{'}\rangle+a_8|7{,}10^{'}\rangle+a_9|83^{'}\rangle+a_{10}|90^{'}\rangle+a_{11}|10{,}1^{'}\rangle+a_{12}|11{,}2^{'}\rangle$$

where the coefficients $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12})$ take the following 12 group values:

- $(1,1,1,1,1,1,1,1,1,1,1,1)$
- $(1,\omega,\omega^2,\omega^3,1,\omega,\omega^2,\omega^3,1,\omega,\omega^2,\omega^3)$
- $(1,\omega^2,1,\omega^2,1,\omega^2,1,\omega^2,1,\omega^2,1,\omega^2)$
- $(1,\omega^3,\omega^2,\omega,1,\omega^3,\omega^2,\omega,1,\omega^3,\omega^2,\omega)$
- $(\zeta,\zeta,\zeta,\zeta,\zeta^2,\zeta^2,\zeta^2,\zeta^2,1,1,1,1)$
- $(\zeta,\zeta\omega,\zeta\omega^2,\zeta\omega^3,\zeta^2,\zeta^2\omega,\zeta^2\omega^2,\zeta^2\omega^3,1,\omega,\omega^2,\omega^3)$
- $(\zeta,\zeta\omega^2,\zeta,\zeta\omega^2,\zeta^2,\zeta^2\omega^2,\zeta^2,\zeta^2\omega^2,1,\omega^2,1,\omega^2)$
- $(\zeta,\zeta\omega^3,\zeta\omega^2,\zeta\omega,\zeta^2,\zeta^2\omega^3,\zeta^2\omega^2,\zeta^2\omega,1,\omega^3,\omega^2,\omega)$
- $(\zeta^2,\zeta^2,\zeta^2,\zeta^2,\zeta,\zeta,\zeta,\zeta,1,1,1,1)$
- $(\zeta^2,\zeta^2\omega,\zeta^2\omega^2,\zeta^2\omega^3,\zeta,\zeta\omega,\zeta\omega^2,\zeta\omega^3,1,\omega,\omega^2,\omega^3)$
- $(\zeta^2,\zeta^2\omega^2,\zeta^2,\zeta^2\omega^2,\zeta,\zeta\omega^2,\zeta,\zeta\omega^2,1,\omega^2,1,\omega^2)$
- $(\zeta^2,\zeta^2\omega^3,\zeta^2\omega^2,\zeta^2\omega,\zeta,\zeta\omega^3,\zeta\omega^2,\zeta\omega,1,\omega^3,\omega^2,\omega)$

The second 48 states are as follows:

$$\phi_{b1}=b_1|08^{'}\rangle+b_2|19^{'}\rangle+b_3|2{,}10^{'}\rangle+b_4|3{,}11^{'}\rangle+b_5|40^{'}\rangle+b_6|51^{'}\rangle+b_7|62^{'}\rangle+b_8|73^{'}\rangle+b_9|84^{'}\rangle+b_{10}|95^{'}\rangle+b_{11}|10{,}6^{'}\rangle+b_{12}|11{,}7^{'}\rangle$$

$$\phi_{b2}=b_1|09^{'}\rangle+b_2|1{,}10^{'}\rangle+b_3|2{,}11^{'}\rangle+b_4|38^{'}\rangle+b_5|41^{'}\rangle+b_6|52^{'}\rangle+b_7|63^{'}\rangle+b_8|70^{'}\rangle+b_9|85^{'}\rangle+b_{10}|96^{'}\rangle+b_{11}|10{,}7^{'}\rangle+b_{12}|11{,}4^{'}\rangle$$

$$\phi_{b3}=b_1|0{,}10^{'}\rangle+b_2|1{,}11^{'}\rangle+b_3|28^{'}\rangle+b_4|39^{'}\rangle+b_5|42^{'}\rangle+b_6|53^{'}\rangle+b_7|60^{'}\rangle+b_8|71^{'}\rangle+b_9|86^{'}\rangle+b_{10}|97^{'}\rangle+b_{11}|10{,}4^{'}\rangle+b_{12}|11{,}5^{'}\rangle$$

$$\phi_{b4}=b_1|0{,}11^{'}\rangle+b_2|18^{'}\rangle+b_3|29^{'}\rangle+b_4|3{,}10^{'}\rangle+b_5|43^{'}\rangle+b_6|50^{'}\rangle+b_7|61^{'}\rangle+b_8|72^{'}\rangle+b_9|87^{'}\rangle+b_{10}|94^{'}\rangle+b_{11}|10{,}5^{'}\rangle+b_{12}|11{,}6^{'}\rangle$$

where the coefficients $(b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10},b_{11},b_{12})$ take the following 12 group values:

$$\bullet(1,1,1,1,1,1,1,1,1,1,1,1)$$
$$\bullet(1,\omega,\omega^2,\omega^3,1,\omega,\omega^2,\omega^3,1,\omega,\omega^2,\omega^3)$$
$$\bullet(1,\omega^2,1,\omega^2,1,\omega^2,1,\omega^2,1,\omega^2,1,\omega^2)$$
$$\bullet(1,\omega^3,\omega^2,\omega,1,\omega^3,\omega^2,\omega,1,\omega^3,\omega^2,\omega)$$
$$\bullet(\zeta^2,\zeta^2,\zeta^2,\zeta^2,1,1,1,1,\zeta,\zeta,\zeta,\zeta)$$
$$\bullet(\zeta^2,\zeta^2\omega,\zeta^2\omega^2,\zeta^2\omega^3,1,\omega,\omega^2,\omega^3,\zeta,\zeta\omega,\zeta\omega^2,\zeta\omega^3)$$
$$\bullet(\zeta^2,\zeta^2\omega^2,\zeta^2,\zeta^2\omega^2,1,\omega^2,1,\omega^2,\zeta,\zeta\omega^2,\zeta,\zeta\omega^2)$$
$$\bullet(\zeta^2,\zeta^2\omega^3,\zeta^2\omega^2,\zeta^2\omega,1,\omega^3,\omega^2,\omega,\zeta,\zeta\omega^3,\zeta\omega^2,\zeta\omega)$$
$$\bullet(\zeta,\zeta,\zeta,\zeta,1,1,1,1,\zeta^2,\zeta^2,\zeta^2,\zeta^2)$$
$$\bullet(\zeta,\zeta\omega,\zeta\omega^2,\zeta\omega^3,1,\omega,\omega^2,\omega^3,\zeta^2,\zeta^2\omega,\zeta^2\omega^2,\zeta^2\omega^3)$$
$$\bullet(\zeta,\zeta\omega^2,\zeta,\zeta\omega^2,1,\omega^2,1,\omega^2,\zeta^2,\zeta^2\omega^2,\zeta^2,\zeta^2\omega^2)$$
$$\bullet(\zeta,\zeta\omega^3,\zeta\omega^2,\zeta\omega,1,\omega^3,\omega^2,\omega,\zeta^2,\zeta^2\omega^3,\zeta^2\omega^2,\zeta^2\omega)$$

Next,we present the corresponding matrices $\{U_n^i,\ i=0,1,2,\ n=1,2,\cdots 11,12\}$ to give 36 maximally entangled statesin $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$:

Let $U_n, n=1,2,\cdots,11,12$ be the set of unitary matrices that give rise to the 12 member UMEB in $\mathbb{C}^4\otimes\mathbb{C}^4$, where

$$U_1=\frac{1}{\sqrt{2}}\sigma_x\otimes(\sigma_x-\sigma_y),$$
$$U_2=\frac{1}{\sqrt{2}}(\sigma_x-\sigma_y)\otimes\sigma_z,$$
$$U_3=\frac{1}{\sqrt{2}}\sigma_z\otimes(\sigma_z-\sigma_y),$$
$$U_4=\frac{1}{\sqrt{2}}(\sigma_z-\sigma_y)\otimes\sigma_x,$$
$$U_5=\frac{1}{3}(\sigma_x+\sigma_y+\sigma_z)\otimes(\sigma_x+\sigma_y+\sigma_z),$$
$$\{U_6,\cdots,U_{12}\}=\{I\otimes I, I\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z, \sigma_x\otimes I, \sigma_y\otimes I, \sigma_z\otimes I)$$

Denote $U_n^i=W^i\otimes U_n,\ i=0,1,2;\ n=1,2,\cdots,6$, through directly calculation, we have:

$$U_n^0=\begin{pmatrix}U_n & 0_3 & 0_3\\ 0_3 & U_n & 0_3\\ 0_3 & 0_3 & U_n\end{pmatrix},\quad U_n^1=\begin{pmatrix}U_n & 0_3 & 0_3\\ 0_3 & \zeta U_n & 0_3\\ 0_3 & 0_3 & \zeta^2 U_n\end{pmatrix},\quad U_n^2=\begin{pmatrix}U_n & 0_3 & 0_3\\ 0_3 & \zeta^2 U_n & 0_3\\ 0_3 & 0_3 & \zeta U_n\end{pmatrix}$$

$$|\psi_n^i\rangle=(I\otimes U_n^i)\frac{1}{2\sqrt{3}}\sum_{\alpha=0}^{11}|\alpha\rangle\otimes|\alpha\rangle,\ i=0,1,2;\ n=1,2,\cdots,12 \quad (4)$$

then the following 36 states $\{|\psi_n^i\rangle\}$ are all orthonormal maximally entangled states in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$. It is easy to verified that all the above 132 states in (3) and (4) construct a UMEB in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$.

4.3. UMEB in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$ from that in $\mathbb{C}^6\otimes\mathbb{C}^6$

We first present the corresponding matrices $\{U_{nm}^{ij}, i=0,1,\ j=1,m,n=0,1,2,\cdots,5\}$ to give 72 maximally entangled statesin $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$:

Now $\omega = e^{\frac{2\pi\sqrt{-1}}{6}}$, $U_{nm}=\sum_{k=0}^{5} e^{\frac{2\pi\sqrt{-1}}{6}kn} |k\oplus m\rangle\langle k|,\ m,n=0,1,2,3,4,5$ and

$$W=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\quad S=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Denote $W^k = diag(w_{k+1,1}, w_{k+1,2}),\ k=0,1$, we have $W^0 = diag(1,1),\ W^1 = diag(1,-1)$, then we have the following 72 orthonormal maximally entangled states in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$:

Define

$$|\phi_{nm}^{ij}\rangle = \frac{1}{\sqrt{12}}\sum_{i=0}^{11}|i'\rangle\otimes(U_{nm}^{ij}|i'\rangle) = \frac{1}{2\sqrt{3}}\sum_{i=0}^{11}|i\rangle\otimes((W^k S^j\otimes U_{nm})|i'\rangle) \qquad (5)$$

Now we illustrate the above 72 states. The 72 states are as follows:

$\phi_1 = a_1|06'\rangle + a_2|17'\rangle + a_3|28'\rangle + a_4|39'\rangle + a_5|4{,}10'\rangle + a_6|5{,}11'\rangle + a_7|60'\rangle + a_8|71'\rangle + a_9|82'\rangle + a_{10}|93'\rangle + a_{11}|10{,}4'\rangle + a_{12}|11{,}5'\rangle$

$\phi_2 = a_1|07'\rangle + a_2|18'\rangle + a_3|29'\rangle + a_4|3{,}10'\rangle + a_5|4{,}11'\rangle + a_6|56'\rangle + a_7|61'\rangle + a_8|72'\rangle + a_9|83'\rangle + a_{10}|94'\rangle + a_{11}|10{,}5'\rangle + a_{12}|11{,}0'\rangle$

$\phi_3 = a_1|08'\rangle + a_2|19'\rangle + a_3|2{,}10'\rangle + a_4|3{,}11'\rangle + a_5|46'\rangle + a_6|57'\rangle + a_7|62'\rangle + a_8|73'\rangle + a_9|84'\rangle + a_{10}|95'\rangle + a_{11}|10{,}0'\rangle + a_{12}|11{,}1'\rangle$

$\phi_4 = a_1|09'\rangle + a_2|1{,}10'\rangle + a_3|2{,}11'\rangle + a_4|36'\rangle + a_5|47'\rangle + a_6|58'\rangle + a_7|63'\rangle + a_8|74'\rangle + a_9|85'\rangle + a_{10}|90'\rangle + a_{11}|10{,}1'\rangle + a_{12}|11{,}2'\rangle$

$\phi_5 = a_1|0{,}10'\rangle + a_2|1{,}11'\rangle + a_3|26'\rangle + a_4|37'\rangle + a_5|48'\rangle + a_6|59'\rangle + a_7|64'\rangle + a_8|75'\rangle + a_9|80'\rangle + a_{10}|91'\rangle + a_{11}|10{,}2'\rangle + a_{12}|11{,}3'\rangle$

$\phi_6 = a_1|0{,}11'\rangle + a_2|16'\rangle + a_3|27'\rangle + a_4|38'\rangle + a_5|49'\rangle + a_6|5{,}10'\rangle + a_7|65'\rangle + a_8|70'\rangle + a_9|81'\rangle + a_{10}|92'\rangle + a_{11}|10{,}3'\rangle + a_{12}|11{,}4'\rangle$

where the coefficient $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12})$ take the following 12 group values:

- $(1,1,1,1,1,1,1,1,1,1,1,1)$
- $(1,\omega,\omega^2,\omega^3,\omega^4,\omega^5,1,\omega,\omega^2,\omega^3,\omega^4,\omega^5)$
- $(1,\omega^2,\omega^4,1,\omega^2,\omega^4,1,\omega^2,\omega^4,1,\omega^2,\omega^4)$
- $(1,\omega^3,1,\omega^3,1,\omega^3,1,\omega^3,1,\omega^3,1,\omega^3)$
- $(1,\omega^4,\omega^2,1,\omega^4,\omega^2,1,\omega^4,\omega^2,1,\omega^4,\omega^2)$
- $(1,\omega^5,\omega^4,\omega^3,\omega^2,\omega,1,\omega^5,\omega^4,\omega^3,\omega^2,\omega)$
- $(-1,-1,-1,-1,-1,-1,1,1,1,1,1,1)$
- $(-1,-\omega,-\omega^2,-\omega^3,-\omega^4,-\omega^5,1,\omega,\omega^2,\omega^3,\omega^4,\omega^5)$
- $(-1,-\omega^2,-\omega^4,-1,-\omega^2,-\omega^4,1,\omega^2,\omega^4,1,\omega^2,\omega^4)$
- $(-1,-\omega^3,-1,-\omega^3,-1,-\omega^3,1,\omega^3,1,\omega^3,1,\omega^3)$
- $(-1,-\omega^4,-\omega^2,-1,-\omega^4,-\omega^2,1,\omega^4,\omega^2,1,\omega^4,\omega^2)$
- $(-1,-\omega^5,-\omega^4,-\omega^3,-\omega^2,-\omega,1,\omega^5,\omega^4,\omega^3,\omega^2,\omega)$

Next, we present the corresponding matrices $\{U_n^i, i=0,1, n=1,2,\cdots 29,30\}$ to give 60 maximally entangled statesin $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$:

Let $\{U_n, n=1,2,\cdots,29,30\}$ be the set of unitary matrices that give rise to the 30-member UMEB in $\mathbb{C}^6\otimes\mathbb{C}^6$.Denote $U_n^i = W^i\otimes U_n,\ i=0,1,\ n=1,2,\cdots 29,30$, then we have the following 60 orthonormal maximally entangled states in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$:

The first 36 states are as follows:

$\phi_1 = a_1|03'\rangle + a_2|14'\rangle + a_3|25'\rangle + a_4|30'\rangle + a_5|41'\rangle + a_6|52'\rangle + a_7|69'\rangle + a_8|7{,}10'\rangle + a_9|8{,}11'\rangle + a_{10}|96'\rangle + a_{11}|10{,}7'\rangle + a_{12}|11{,}8'\rangle$

$\phi_2 = a_1|04'\rangle + a_2|15'\rangle + a_3|23'\rangle + a_4|31'\rangle + a_5|42'\rangle + a_6|50'\rangle + a_7|6{,}10'\rangle + a_8|7{,}11'\rangle + a_9|89'\rangle + a_{10}|97'\rangle + a_{11}|10{,}8'\rangle + a_{12}|11{,}6'\rangle$

$\phi_3 = a_1|05'\rangle + a_2|13'\rangle + a_3|24'\rangle + a_4|32'\rangle + a_5|40'\rangle + a_6|51'\rangle + a_7|6{,}11'\rangle + a_8|79'\rangle + a_9|8{,}10'\rangle + a_{10}|98'\rangle + a_{11}|10{,}6'\rangle + a_{12}|11{,}7'\rangle$

where the coefficients $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12})$ take the following 12 group values:

$$\bullet(1,1,1,1,1,1,1,1,1,1,1,1)$$
$$\bullet(1,\omega^2,\omega,1,\omega^2,\omega,1,\omega^2,\omega,1,\omega^2,\omega)$$
$$\bullet(1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2)$$
$$\bullet(-1,-1,-1,1,1,1,-1,-1,-1,1,1,1)$$
$$\bullet(-1,-\omega^2,-\omega,1,\omega^2,\omega,-1,-\omega^2,-\omega,1,\omega^2,\omega)$$
$$\bullet(-1,-\omega,-\omega^2,1,\omega,\omega^2,-1,-\omega,-\omega^2,1,\omega,\omega^2)$$
$$\bullet(1,1,1,1,1,1,-1,-1,-1,-1,-1,-1)$$
$$\bullet(1,\omega^2,\omega,1,\omega^2,\omega,-1,-\omega^2,-\omega,-1,-\omega^2,-\omega)$$
$$\bullet(1,\omega,\omega^2,1,\omega,\omega^2,-1,-\omega,-\omega^2,-1,-\omega,-\omega^2)$$
$$\bullet(-1,-1,-1,1,1,1,1,1,1,-1,-1,-1)$$
$$\bullet(-1,-\omega^2,-\omega,1,\omega^2,\omega,1,\omega^2,\omega,-1,-\omega^2,-\omega)$$
$$\bullet(-1,-\omega,-\omega^2,1,\omega,\omega^2,1,\omega,\omega^2,-1,-\omega,-\omega^2)$$

The second 24 states are as follows:

Through directly calculation, we have：

$$U_n^i = I_{12}^i - (1-e^{i\theta})(|\psi_j\rangle\langle\psi_j| + |\psi_{j+6}\rangle\langle\psi_{j+6}| + |\psi_{j+12}\rangle\langle\psi_{j+12}| + |\psi_{j+18}\rangle\langle\psi_{j+18}|)$$

$i = 0,1,2,3,\ j = 1,2,3,4,5,6$, and

$$|\psi_{1,2}\rangle = \frac{1}{N}(|0\rangle \pm \phi|1\rangle), |\psi_{3,4}\rangle = \frac{1}{N}(|1\rangle \pm \phi|2\rangle), |\psi_{5,6}\rangle = \frac{1}{N}(|2\rangle \pm \phi|0\rangle)$$

$$|\psi_{7,8}\rangle = \frac{1}{N}(|3\rangle \pm \phi|4\rangle), |\psi_{9,10}\rangle = \frac{1}{N}(|4\rangle \pm \phi|5\rangle), |\psi_{11,12}\rangle = \frac{1}{N}(|5\rangle \pm \phi|3\rangle)$$

$$|\psi_{13,14}\rangle = \frac{1}{N}(|6\rangle \pm \phi|7\rangle), |\psi_{15,16}\rangle = \frac{1}{N}(|7\rangle \pm \phi|8\rangle), |\psi_{17,18}\rangle = \frac{1}{N}(|8\rangle \pm \phi|6\rangle)$$

$$|\psi_{19,20}\rangle = \frac{1}{N}(|9\rangle \pm \phi|10\rangle), |\psi_{21,22}\rangle = \frac{1}{N}(|10\rangle \pm \phi|11\rangle), |\psi_{23,24}\rangle = \frac{1}{N}(|11\rangle \pm \phi|9\rangle)$$

$$I_{12}^0 = \begin{pmatrix} I_3 & 0_3 & 0_3 & 0_3 \\ 0_3 & I_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & I_3 & 0_3 \\ 0_3 & 0_3 & 0_3 & I_3 \end{pmatrix}, \quad I_{12}^1 = \begin{pmatrix} I_3 & 0_3 & 0_3 & 0_3 \\ 0_3 & -I_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & I_3 & 0_3 \\ 0_3 & 0_3 & 0_3 & -I_3 \end{pmatrix}$$

$$I_{12}^2 = \begin{pmatrix} I_3 & 0_3 & 0_3 & 0_3 \\ 0_3 & I_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & -I_3 & 0_3 \\ 0_3 & 0_3 & 0_3 & -I_3 \end{pmatrix}, \quad I_{12}^3 = \begin{pmatrix} I_3 & 0_3 & 0_3 & 0_3 \\ 0_3 & -I_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & -I_3 & 0_3 \\ 0_3 & 0_3 & 0_3 & I_3 \end{pmatrix}$$

Define

$$|\psi_n^i\rangle = (I \otimes U_n^i)\frac{1}{2\sqrt{3}}\sum_{\alpha=0}^{11}|\alpha\rangle \otimes |\alpha\rangle,\ i = 0,1,2,3; n = 1,2,\cdots 5,6 \qquad (6)$$

then the above 24 states $\{|\psi_n^i\rangle\}$ are all orthonormal maximally entangled states in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$. It is easy to verified that all the above 132 states in (5) and (6) construct a UMEB in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$.

There are three ways to construct UMEBs from the UMEBs of $\mathbb{C}^{3}\otimes\mathbb{C}^{3}$,$\mathbb{C}^{4}\otimes\mathbb{C}^{4}$ and $\mathbb{C}^{6}\otimes\mathbb{C}^{6}$.Each way shows 132-member UMEB in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$.If the 132 unitary matrices of the two methods are similar to each other, we say that the two methods are the same.Because it is too complicated to judge whether the unitary matrices of each method are similar one by one, we give the eigenvalues of some special unitary matrices of the three construction methods as follows:

Table 1:Order of eigenvalues of UMEBs in $\mathbb{C}^{12}\otimes\mathbb{C}^{12}$

	$O_{\min}(U_n^i)$	$O_{\max}(U_n^i)$	$O_{\min}(U_{mn}^{ij})$	$O_{\max}(U_{mn}^{ij})$
$\mathbb{C}^{3}\otimes\mathbb{C}^{3}$	1	∞	1	12
$\mathbb{C}^{4}\otimes\mathbb{C}^{4}$	1	12	1	12
$\mathbb{C}^{6}\otimes\mathbb{C}^{6}$	1	∞	1	12

From the reference[5], we can know that at least two of the three UMEBs obtained in this way are not equivalent. From Table 1, we can know that the ways from dimension 3,4 are different and the ways from 4,6 are different. Since the 30-member UMEBs of $\mathbb{C}^{6}\otimes\mathbb{C}^{6}$ are constructed from $\mathbb{C}^{3}\otimes\mathbb{C}^{3}$, there is a certain similarity from dimension 3,6. If we want to judge whether the two constructions are completely consistent, we need to further determine whether each pair of unitary matrices is completely similar. If there are one or more pairs of unitary matrices are not similar, then we can conclude that the ways from dimension 3,6 are different.

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