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Probabilists' Hermite Collocation Method for Approximating Second Order Linear Boundary Value Problems in Ordinary Differential Equations

Tersoo Luga[1], Vershima Benjamin Iyorter[2] and Simon Sunday Isah[3]
[1, 3] *Department of Mathematics/Statistics/Computer Science, College of Science,*
Federal University of Agriculture, Makurdi, Nigeria
[2] *Department of Mathematics/Computer Science, College of Natural and Applied Sciences,*
University of Mkar, Mkar, Nigeria
Correspondence Author: TersooLuga

Abstract:*A collocation method for approximating second order boundary value problems in ordinary differential equations was developed using the probabilists' Hermite polynomial of degree eight as basis function. Five examples were carefully chosen to include homogeneous and nonhomogeneous boundary value problems with constant and variable coefficients. The boundary value problems involve Dirichlet, Neumann and Robin boundary conditions. The collocation method when applied to the boundary value problems providedgood approximations of the analytical solutions. However, the boundary value problems with Dirichlet boundary condition gave a better approximation of the analytical solution as compared to the boundary value problems with Neumann and Robin conditions. We also observed that the accuracy of the collocation method increased as more terms of the probabilists' Hermite polynomial were used as basis function.*

Key words: *Collocation, Probabilists' Hermite Polynomial, Physicists' Hermite polynomial.*

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I. Introduction

Boundary value problems (BVPs) in ordinary differential equations (ODEs) are used to model many physical phenomena in engineering, sciences especially physics and other related areas. Some examples include spring problem electrical problem, buoyance problem, astronomy, biology, boundary layer theory, heat transfer, diffusion process, electromagnetism, deflection in cables, Sturm-Liouville problem among others [1-3] etc.

Although there are many applications of BVPs and numerous analytical solutions have been developed for solving them, many do not have an exact solution expressed in terms of elementary functions such as polynomial functions, trigonometric functions logarithmic functions etc. or their combinations. In many instances, even if a differential equation can be solved analytically, sound mathematical theory is often required and the closed form solution may be too complicated [4]. In such situations approximated solutions are preferred. Li *et al.* [4] discussed two types of approaches for providing approximate solutions to differential equations viz: semi-analytical methods and numerical methods. Semi-analytical methods are those methods in which the approximate solutions are expressed in terms of simpler functions such as series solution, perturbation techniques or asymptotical methods. On the other hand, numerical solutions are given in terms of discrete numerical values to a certain accuracy. In recent times, the number of arrays and associated tables or plots are obtained using computer programmes which provide effective solutions to many problems that were impossible to obtain before.

There are four traditional approximate methods for solving BVPs, namely, the finite difference method, shooting method, collocation method and finite element method [5]. In this work, the collocation approach using probabilists' Hermite polynomial as basis function is constructed and applied to approximate some BVPs. A collocation method is a method which involves the determination of an approximation solution to an equation using a suitable set of functions, sometimes called trial or basis functions. The approximate solution is required to satisfy the governing equation and its supplementary conditions at certain points in the range of interest called collocation points.

The basis function used for developing a collocation method may be a monomial, polynomial function [5], spline function [6] etc. polynomials are susceptible to Runge phenomenon and on the other hand, monomial elements are not orthogonal to one another as functions, to this end, using many terms of such functions make the coefficient matrix of the linear equations ill-conditioned when large. The use of orthogonal polynomials as basis functions is preferred [5].

Because of the availability of computers, many researchers in recent times have applied orthogonal polynomials as basis functions for developing approximate methods which are comparable to the analytical solutions or existing methods. For instance, the Chebyshev polynomials [7], Legendre polynomial [8, 9], Jacobi polynomial [10] among others have been used as basis functions to develop collocation methods for approximating ODEs with accurate numerical solutions. Similarly, Splines have been effectively utilized as basis functions for constructing collocation methods for solving BVPs [6, 11-13].

The Hermite polynomials are orthogonal polynomials that have been applied to solve different forms of ordinary differential equations and even some partial differential equations. Aboiyar*et al.* [14] used Hermite polynomials to construct linear multistep methods for approximating initial value problems. Similarly, [15] developed a collocation method for solving differential difference equations using Hermite polynomial as a basis function. Nagaigh and Kumur [16] approximated the solution of a second other partial differential equation using the Hermite polynomial as basis functions. Other works on collocation methods using Hermite polynomials as basis function include: A new decoupling technique for the Hermite cubic collocation equations arising from boundary value problems [17], Geometric Hermite interpolation by cubic splines [18], Asymptotic convergence of cubic Hermite collocation method for axial dispersion model [19] Cubic Hermite collocation method for solving boundary value problems with Dirichlet, Neumann and Robin conditions [3], etc. Other similar papers on collocation methods using Hermite polynomials as basis functions can be found in literature.

There are two types of Hermite polynomials with different variances, namely: Physicists' Hermite polynomial and Probabilits' Hermite polynomial [20]. However, we have observed that almost all the collocation methods based on Hermite polynomials in the literature we have been able to collect make use of the Physicists' Hermite polynomial as the basis function. The few papers within our reach on Probabilits' polynomials are on initial value problems of ordinary differential equations, see [15, 21]. In this work, the Probabilist's Hermite polynomial of degree 8 is used as basis function to develop a collocation method for approximating second order linear boundary value problems of ordinary differential equations with Dirichlet, Neumann and Robin boundary conditions.

The paper is divided into 5 Sections: Section 1 deals with the general introduction as presented above, Section 2 gives the description of the developed collocation method using the probabilits' Hermite polynomial of degree 8. Results are presented in Section 3, while the discussion of results and conclusion are done in Section 4.

II. Methods

This Sections explains how the probabilists' Hermit polynomial is used to developed a collocation method for approximating second order boundary value problems in ordinary differential equations.

Second Order Boundary Value Problems (BVPs)

Let x and y denote the independent and dependent variables respectively, a second order boundary value problem is defined by

$$y''(x) = f\big(x, y(x), y'(x)\big), \quad a \le x \le b \tag{3.1}$$

with the boundary conditions

$$\left.\begin{aligned} \alpha_1 y_1(a) + \alpha_2 y_1'(a) = \gamma_1 \\ \beta_1 y_2(b) + \beta_2 y_2'(b) = \gamma_2 \end{aligned}\right\}, \tag{3.2}$$

where $\alpha_1, \alpha_2, \beta_1,$ and β_2 are constants. $\alpha_1$andα_2 are not both zero and a similar situation holds for β_1, and β_2. If α_2, and β_2 are both equal to zero in equation (3.2), then equation (3.1) is said to have Dirichlet boundary condition at both boundaries. Equation (3.1) has Neumann boundary condition at both ends if both $\alpha_1$and β_1 are equal to zero. Equation (3.1) is said to have Robin boundary condition at both boundaries if none of the constants $\alpha_1, \alpha_2, \beta_1,$ and $\beta_2$is equal to zero. We may have situations where both boundary conditions may have Dirichlet, Neumann or Robin. However, we may have some cases where the boundary conditions may be different at the end points.

The Probabilists' Hermit Polynomial

The probabilists' Hermit polynomial is generated from the formula

$$H_n(x) = (-1)^n exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}exp\left(-\frac{x^2}{2}\right) = \left(x - \frac{d}{dx}\right)^n \cdot 1 \tag{3.3}$$

The recursive formula of equation (3.3) is given by

$$H_{n+1}(x) = xH_n(x) - H_n'(x) \tag{3.4}$$

The first term H_0 is generated from equation (3.3) while the rest of the other terms can be generated conveniently from equation (3.4). The first eleven terms of the probabilists' polynomial generated as explained above is given below

$$\left.\begin{array}{l} H_0 = 1 \\ H_1 = x \\ H_2 = x^2 - 1 \\ H_3 = x^3 - 3x \\ H_4 = x^4 - 6x^2 \\ H_5 = x^5 - 10x^3 + 15 \\ H_6 = x^6 - 15x^4 + 45x^2 - 15 \\ H_7 = x^7 - 21x^5 + 105x^3 - 105x \\ H_8 = x^8 - 28x^6 + 210x^4 - 420x^2 + 105 \\ H_9 = x^9 - 36x^7 + 378x^5 - 420x^3 + 105x \\ H_{10} = x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945 \end{array}\right\} \tag{3.5}$$

Collocation Method for Approximating Boundary Value Problems

In a collocation method, the idea is to reduce a boundary value problem to a set of solvable algebraic equations. The set of basis function, $\varphi_j(x), 1 \le j \le N$, used to obtain approximate solution may be a monomial, polynomial, trigonometric function, spline or other simple function [5].

To solve a boundary value problem using a collocation method, we make an assumption that the solution of the boundary value problem can be approximated using

$$y(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \cdots + a_N\varphi_N(x) \tag{3.6}$$

or

$$y(x) = \sum_{j=1}^{N} a_j \varphi_j\,(x) \tag{3.7}$$

where N is the number of terms of a basis or trial function.

Collocation Method using Probabilist's Hermit Polynomials as Basis Functions

To construct a collocation method for approximating boundary value problems using the probabilists' polynomial as basis function, equation (3.7) is redefined as

$$y(x) = \sum_{j=1}^{N} a_j H_j\,(x). \tag{3.8}$$

The first eleven terms of $H_j(x)$ are given in equation (3.5). To apply equation (3.8) to approximate a boundary value problem, the first and the last term corresponding to $j = 1$ and $j = N$ are boundary conditions, the remaining $N - 2$ equations are obtained from the given equation by differentiating equation (3.8) the required number of times and then evaluating x_i for $2 \le i \le N - 1$, where x_i is as defined in [5] as shown below

$$x_i = a + \frac{i-1}{N-1}(b-a) \tag{3.9}$$

Solving Boundary Value Problem with Probabilists' Hermite Polynomials

To approximate the solution of BVP using a collocation method with Probabilists' Hermite polynomial as basis function involves two main stages namely, finding solution at the boundary points and finding solution at the interior mesh points.

Solution at the Boundary Mesh Points

Three cases are considered

Dirichlet Boundary Condition

For a BVP with Dirichlet boundary condition at both ends, we get the following

$$y(a) = \sum_{j=1}^{N} a_j H_j\,(a) \tag{3.10}$$

and

$$y(b) = \sum_{j=1}^{N} a_j H_j\,(b). \tag{3.11}$$

Dirichlet Boundary Condition

A BVP with Neumann condition at both ends takes the form

$$y'(a) = \sum_{j=1}^{N} a_j H_j'(a) \tag{3.12}$$

and

$$y'(b) = \sum_{j=1}^{N} a_j H_j'(b) \tag{3.13}$$

Robin Boundary Condition

Robin condition given at both end of a BVP takes the form

$$y(a) + y'(a) = \sum_{j=1}^{N} a_j H_j(a) + \sum_{j=1}^{N} a_j H_j'(a) \tag{3.14}$$

and

$$y(b) + y'(b) = \sum_{j=1}^{N} a_j H_j(b) + \sum_{j=1}^{N} a_j H_j'(b) \tag{3.15}$$

Solution at the Interior Mesh Points

Consider equation (3.8), which is the approximate solution, the first and last points corresponding to $j = 1$ and $j = N$ are boundary points. The discretization at these points is as explained in equations $(3.10) - (3.15)$. The remaining $N - 2$ equations are obtained from the differential equation evaluated at x_i .

Consider the linear form of equation (3.1) denoted by

$$P(x)y''(x) + Q(x)y'(x) + R(x)y = G(x). \tag{3.16}$$

Finding the first and second derivatives of equation (3.8) yields

$$y'(x) = \sum_{j=1}^{N} a_j H_j'(x) \tag{3.17}$$

and

$$y''(x) = \sum_{j=1}^{N} a_j H_j''(x) \tag{3.18}$$

respectively.

The $N - 2$ equations on the interior mesh points are obtained from

$$\sum_{j=1}^{N} a_j \left(P(x_i)H_j''(x_i) + Q(x_i)H_j'(x_i) + R(x_i)H_j(x_i)\right) = G(x_i) \tag{3.19}$$

where x_i is defined by equation (3.9), [5].

Once the discretization on the boundary points and the interior points are obtained, we have an $N \times N$ matrix of the form

$$\begin{pmatrix} \boldsymbol{c_{11}} & \boldsymbol{c_{12}} & \cdots & \boldsymbol{c_{1N}} \\ \boldsymbol{c_{21}} & \boldsymbol{c_{22}} & \cdots & \boldsymbol{c_{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{c_{N1}} & \boldsymbol{c_{N2}} & \cdots & \boldsymbol{c_{NN}} \end{pmatrix} \begin{pmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_N} \end{pmatrix} = \begin{pmatrix} \boldsymbol{k_1} \\ \boldsymbol{k_2} \\ \vdots \\ \boldsymbol{k_N} \end{pmatrix} \tag{3.20}$$

where the coefficient matrix, C_{ij} and k_i are constants. C_{ij} is defined by

$$C_{ij} = \begin{cases} \displaystyle\sum_{j=1}^{N} H_j(a), \text{or} \sum_{j=1}^{N} H_j'(a) \text{ or} \sum_{j=1}^{N} H_j(a) + \sum_{j=1}^{N} H_j'(a), & i = 1 \\ \displaystyle\sum_{j=1}^{N} \left(P(x_i)H_j''(x_i) + Q(x_i)H_j'(x_i) + R(x_i)H_j(x_i)\right) & i = 2 \; to \; N - 1 \\ \displaystyle\sum_{j=1}^{N} H_j(b), \text{or} \sum_{j=1}^{N} H_j'(b) \text{ or} \sum_{j=1}^{N} H_j(b) + \sum_{j=1}^{N} H_j'(b) & i = N \end{cases}$$

and

$$k_i = \begin{cases} \gamma_1, & i = 1 \\ G(x_i), & i = 2 \; to \; N - 1 \\ \gamma_2, & i = N. \end{cases}$$

Equation (3.20) is solved to get the values of $a_j, 1 \leq j \leq N$. These values are substituted in equation (3.8) to get the required approximate solution inform of series. Note that N is the truncated number of terms of the probabilists' Hermite polynomial which is equivalent to a polynomial of degree $N-1$.

III. Result

The method formulated and described in Section 2 is used to approximate the solutions of some boundary value problems (BVPs) of ordinary differential equations in this Section. The approximate solutions are in form of series solutions and are compared with the analytical solutions at some selected mesh points within the given interval and the results and absolute errors are displayed in Tables. The test problems include homogeneous and nonhomogeneous ordinary differential equations with constant and variable coefficients with the following boundary conditions: (i) Dirichlet, (ii) Neumann and (iii) Robin.

The first test problem and the boundary conditions are obtained from [5], while the remaining four are selected from [23]. The simplification and implementation of the BVPs are carried out with the aid of MAPLE and MATLAB software. The coefficients for Examples 1-5 are provided in the Appendix, these coefficients are substituted into equation (3.8) to get the various required approximate solution in series form.

Example 1

Consider the homogeneous boundary value problem

$y'' - 4y = 0, \qquad 0 \leq x \leq 1.$

$y(0) = 1$ and $y(1) = 3$

The analytical solution is

$y(x) = \frac{3-\exp(-2)}{\exp(2)-\exp(-2)}\exp(2x) + \frac{\exp(2)-3}{\exp(2)-\exp(-2)}\exp(-2x).$

Approximate Solution for Example 1

The solutions for Example 1 which corresponds to probabilists' Hermite polynomial of degree (i) Three (ii) Four (iii) Five (iv) Six (v) Seven and (vi) Eight respectively are provided in Tables $1.1 - 1.6$.

Table 1.1: Analytical and Approximate Solution for Example 1 using Hermite Polynomial of Degree 3

n	x	$y(x_n)$	y_n	$\lvert y(x_n) - y_n \rvert$
0	2.0	0	9.2981e-16	9.2981e-16
1	2.1	1.8609e-02	1.8609e-02	2.0701e-08
2	2.2	3.2536e-02	3.2536e-02	1.5185e-08
3	2.3	4.2048e-02	4.2048e-02	1.1736e-08
4	2.4	4.7368e-02	4.7368e-02	7.8748e-09
5	2.5	4.8684e-02	4.8684e-02	4.1657e-09
6	2.6	4.6154e-02	4.6154e-02	4.7604e-10
7	2.7	3.9912e-02	3.9912e-02	3.3357e-09
8	2.8	3.0075e-02	3.0075e-02	6.7756e-09
9	2.9	1.6742e-02	1.6742e-02	1.1687e-08
10	3.0	0	7.4940e-16	7.4940e-16

Table 1.2: Analytical and Approximate Solution for Example 1 using Hermite Polynomial of Degree 4

n	x	$y(x_n)$	y_n^4	$\lvert y(x_n) - y_n \rvert$
0	0	1.0000e+00	1.0000e+00	0
1	0.1	9.7776e-01	9.7671e-01	1.0418e-03
2	0.2	9.9475e-01	9.9397e-01	7.8126e-04
3	0.3	1.0517e+00	1.0515e+00	2.0449e-04
4	0.4	1.1508e+00	1.1510e+00	2.1869e-04
5	0.5	1.2961e+00	1.2966e+00	4.4318e-04
6	0.6	1.4934e+00	1.4941e+00	6.8889e-04
7	0.7	1.7507e+00	1.7519e+00	1.2068e-03
8	0.8	2.0782e+00	2.0802e+00	1.9608e-03
9	0.9	2.4892e+00	2.4914e+00	2.2139e-03
10	1.0	3.0000e+00	3.0000e+00	0

Table 1.3: Analytical and Approximate Solution for Example 1 using Hermite Polynomial of Degree 5

n	x	$y(x_n)$	y_n^5	$\lvert y(x_n) - y_n \rvert$
0	0	1.0000e+00	1.0000e+00	0
1	0.1	9.7776e-01	9.7809e-01	3.3054e-04
2	0.2	9.9475e-01	9.9505e-01	2.9349e-04
3	0.3	1.0517e+00	1.0519e+00	2.4460e-04
4	0.4	1.1508e+00	1.1510e+00	2.4831e-04
5	0.5	1.2961e+00	1.2964e+00	2.6386e-04
6	0.6	1.4934e+00	1.4937e+00	2.6345e-04
7	0.7	1.7507e+00	1.7510e+00	2.7604e-04
8	0.8	2.0782e+00	2.0786e+00	3.4733e-04
9	0.9	2.4892e+00	2.4896e+00	4.0319e-04
10	1.0	3.0000e+00	3.0000e+00	0

Table 1.4: Analytical and Approximate Solution for Example 1 using Hermite Polynomial of Degree 6

n	x	$y(x_n)$	y_n^6	$\lvert y(x_n) - y_n \rvert$
0	0	1.0000e+00	1.0000e+00	0
1	0.1	9.7776e-01	9.7774e-01	1.7068e-05
2	0.2	9.9475e-01	9.9474e-01	1.0046e-05
3	0.3	1.0517e+00	1.0517e+00	4.6340e-06
4	0.4	1.1508e+00	1.1508e+00	8.2755e-07
5	0.5	1.2961e+00	1.2961e+00	4.1109e-06
6	0.6	1.4934e+00	1.4934e+00	9.1943e-06
7	0.7	1.7507e+00	1.7507e+00	1.3370e-05
8	0.8	2.0782e+00	2.0783e+00	1.9897e-05
9	0.9	2.4892e+00	2.4892e+00	2.8692e-05
10	1.0	3.0000e+00	3.0000e+00	4.4409e-16

Table 1.5: Analytical and Approximate Solution for Example 1 using Hermite Polynomial of Degree 7

n	x	$y(x_n)$	y_n^7	$\lvert y(x_n) - y_n \rvert$
0	0	1.0000e+00	1.0000e+00	4.4409e-16
1	0.1	9.7776e-01	9.7776e-01	3.3734e-06
2	0.2	9.9475e-01	9.9475e-01	2.7428e-06
3	0.3	1.0517e+00	1.0517e+00	2.5800e-06
4	0.4	1.1508e+00	1.1508e+00	2.5158e-06
5	0.5	1.2961e+00	1.2961e+00	2.4770e-06
6	0.6	1.4934e+00	1.4934e+00	2.6315e-06
7	0.7	1.7507e+00	1.7507e+00	2.8116e-06
8	0.8	2.0782e+00	2.0782e+00	3.1023e-06
9	0.9	2.4892e+00	2.4892e+00	3.9155e-06
10	1.0	3.0000e+00	3.0000e+00	4.4409e-16

Table 1.6: Analytical and Approximate Solution for Example 1 using Hermite Polynomial of Degree 8

n	x	$y(x_n)$	y_n^8	$\lvert y(x_n) - y_n \rvert$
0	0	1.0000e+00	1.0000e+00	0
1	0.1	9.7776e-01	9.7776e-01	1.3742e-07
2	0.2	9.9475e-01	9.9475e-01	8.2367e-08
3	0.3	1.0517e+00	1.0517e+00	4.8023e-08
4	0.4	1.1508e+00	1.1508e+00	1.1065e-08
5	0.5	1.2961e+00	1.2961e+00	2.3995e-08
6	0.6	1.4934e+00	1.4934e+00	6.0070e-08
7	0.7	1.7507e+00	1.7507e+00	1.0002e-07
8	0.8	2.0782e+00	2.0782e+00	1.3877e-07
9	0.9	2.4892e+00	2.4892e+00	2.0526e-07
10	1.0	3.0000e+00	3.0000e+00	0

Example 2

Consider the nonhomogeneous boundary value problem,

$y'' + 2y' + y = x, \quad 0 \leq x \leq 1.$

$y(0) = -3$ and $y(1) = -1$

The analytical solution is

$y(x) = x - 2 - \exp(-x) + x\exp(-x)$

The approximate solution using a Hermite polynomial of degree eight is provided in Table 2.

Table 2: Analytical and Approximate Solution for Example 2 using Hermite Polynomial of Degree 8

n	x	$y(x_n)$	y_n	$\lvert y(x_n) - y_n\rvert$
0	0	-3.0000e+00	-3.0000e+00	0
1	0.1	-2.7144e+00	-2.7144e+00	1.6665e-09
2	0.2	-2.4550e+00	-2.4550e+00	4.7663e-10
3	0.3	-2.2186e+00	-2.2186e+00	2.8114e-10
4	0.4	-2.0022e+00	-2.0022e+00	9.5631e-10
5	0.5	-1.8033e+00	-1.8033e+00	1.4817e-09
6	0.6	-1.6195e+00	-1.6195e+00	1.9003e-09
7	0.7	-1.4490e+00	-1.4490e+00	2.2412e-09
8	0.8	-1.2899e+00	-1.2899e+00	2.4429e-09
9	0.9	-1.1407e+00	-1.1407e+00	2.8313e-09
10	1.0	-1.0000e+00	-1.0000e+00	6.6613e-16

Example 3

Consider the nonhomogeneous boundary value problem with variable coefficients,

$y'' = 2x^{-2}y - x^{-1}, \qquad 2 \le x \le 3$

$y(2) = 0$ and $y(3) = 0$

The analytical solution is

$$y(x) = \frac{(19x^2 - 5x^3 - 36)}{38x}$$

The approximate solution using a probabilists' Hermite polynomial of degree eight is displayed in Table 3.

Table 3: Analytical and Approximate Solution for Example 3 using Hermite Polynomial of Degree 8

n	x	$y(x_n)$	y_n	$\lvert y(x_n) - y_n\rvert$
0	2.0	0	9.2981e-16	9.2981e-16
1	2.1	1.8609e-02	1.8609e-02	2.0701e-08
2	2.2	3.2536e-02	3.2536e-02	1.5185e-08
3	2.3	4.2048e-02	4.2048e-02	1.1736e-08
4	2.4	4.7368e-02	4.7368e-02	7.8748e-09
5	2.5	4.8684e-02	4.8684e-02	4.1657e-09
6	2.6	4.6154e-02	4.6154e-02	4.7604e-10
7	2.7	3.9912e-02	3.9912e-02	3.3357e-09
8	2.8	3.0075e-02	3.0075e-02	6.7756e-09
9	2.9	1.6742e-02	1.6742e-02	1.1687e-08
10	3.0	0	7.4940e-16	7.4940e-16

Example 4

Consider the boundary value problem with Neumann boundary conditions

$y'' = y' + 2y, \qquad 0 \le x \le 1$

$y'(0) = 1$ and $y'(1) = \frac{(2\exp(2) + \exp(-1))}{3}$

The analytical solution is

$$y(x) = \frac{(\exp(2x) - \exp(-x))}{3}$$

The approximate solution using a Hermite polynomial of degree eight is given in Table 4.

Table 4: Analytical and Approximate Solution for Example 4 using Hermite PolynomialofDegree 8

n	x	$y(x_n)$	y_n	$\lvert y(x_n) - y_n\rvert$
0	0	0	5.8794e-05	5.8794e-05
1	0.1	1.0552e-01	1.0558e-01	5.6824e-05
2	0.2	2.2436e-01	2.2442e-01	5.4659e-05
3	0.3	3.6043e-01	3.6049e-01	5.3539e-05
4	0.4	5.1841e-01	5.1846e-01	5.3390e-05
5	0.5	7.0392e-01	7.0397e-01	5.4368e-05
6	0.6	9.2377e-01	9.2383e-01	5.6587e-05
7	0.7	1.1862e+00	1.1863e+00	6.0225e-05
8	0.8	1.5012e+00	1.5013e+00	6.6301e-05
9	0.9	1.8810e+00	1.8811e+00	7.6061e-05
10	1.0	2.3404e+00	2.3405e+00	8.4033e-05

Example 5

Consider the nonhomogeneous boundary value problem with Robin's boundary conditions

$y'' = y - 4x\exp(x), 0 \le x \le 1$

$y(0) + y'(0) = -1$ and $y(1) + y'(1) = -\exp(1)$
The analytical solution is
$y(x) = x(1 - x)\exp(x)$
The approximate solution using a Hermite polynomial of degree eight is provided in Table 5.

Table 5: Analytical and Approximate Solution for Example 5 using Hermite Polynomial of Degree 8

n	x	$y(x_n)$	y_n	$\|y(x_n) - y_n\|$
0	0	0	9.0204e-06	9.0204e-06
1	0.1	9.9465e-02	9.9475e-02	9.1768e-06
2	0.2	1.9542e-01	1.9543e-01	9.1768e-06
3	0.3	2.8347e-01	2.8348e-01	9.2607e-06
4	0.4	3.5804e-01	3.5805e-01	9.4271e-06
5	0.5	4.1218e-01	4.1219e-01	9.6924e-06
6	0.6	4.3731e-01	4.3732e-01	1.0053e-05
7	0.7	4.2289e-01	4.2290e-01	1.0512e-05
8	0.8	3.5609e-01	3.5610e-01	1.1248e-05
9	0.9	2.2136e-01	2.2138e-01	1.2436e-05
10	1.0	0	1.2885e-05	1.2885e-05

IV. Discussion

The probabilists' Hermite polynomial of degree 8 was used as basis function to construct a collocation method for approximating second order boundary value problems (BVPs) in ordinary differential equations (ODEs). Five (5) Examples were selected to test the accuracy of the collocation method. Examples 1, 2 and 3 are BVPs with Dirichlet boundary condition at both ends while Examples 4 and 5 have Neumann boundary condition and Robin boundary condition at both ends. Probabilists' Hermite polynomials of degree 3-8 were used to develop the collocation method for approximating Example 1. This is to test whether the accuracy of the collocation method is dependent on increasing the degree of the polynomial used as basis function or otherwise. The rest of the test problems were approximate with a collocation method developed by probabilists' Hermite polynomial of degree 8.

The results in Table 1.1-1.6 for Example 1 which correspond to collocation methods developed with the probabilists' Hermite polynomial of degree 3-8 respectively show a progressively increase in the accuracy of the methods measured in terms of the absolute errors. For Example, the largest absolute error recorded in Table 1.1, 1.2 and 1.6 for collocation method using probabilists' polynomial of degree 3, degree 4 and degree 8 are 1.1584×10^{-2},$2.2139 \times 10^{-2}$and 1.002×10^{-7} respectively. In the same vein, Example 4.2 and 4.3 which also have Dirichlet boundary conditions were approximated using the collocation method developed with probabiblists' Hermite polynomial of degree 8 and accurate results were obtained. The largest absolute errors recorded were 1.4817×10^{-9} and 5185×10^{-8} respectively. Example 4 which has Neumann boundary condition at both ends and Example 5 that has Robin condition at both ends were also approximated using the same collocation method developed with a probabilists' Hermite polynomial of degree 8, however, their largest absolute errors recorded are, 5.3390×10^{-5} and 1.0512×10^{-5} respectively. Althrough the results show some level of accuracy, but it is not as accurate as the results obtained in the cases of Examples 1, 2 and 3 which has only Dirichlet boundary condition at both end points. The reduction in accuracy in Example 4 and 5 may be due to the presence of the Neumann and Robin boundary conditions.

V. Conclusion

The collocation method developed using probabilists' Hermite polynomial of degree 8 as basis function is capable of approximating second order linear BVPs having Dirichlet, Neuman and Robin boundary conditions. The collocation method developed is more efficient in terms of accuracy for approximating BVPs having Dirichlet boundary condition. Thus, it can be applied to approximate many models in sciences and engineering especially physics that are in form of second order BVPs with Dirichlet boundary condition. Although the accuracy of the second order BVPs with Neumann and Robins boundary conditions solved in this work is less than BVPs with Dirichlet condition, however, the accuracy of the collocation method derived is sufficient to approximate linear boundary value problems with both Neumann and Robin boundary conditions. To get a better accuracy, higher order probabalists' Hermite polynomial can be used as basis function to construct higher order collocation methods.

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Appendix

Table A1: Coefficients of the Probabilists' Hermite Polynomial of Degree 3-8 for Example 1

DP	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
3	$\frac{817}{403}$	$\frac{1328}{403}$	$\frac{414}{403}$	$\frac{36}{31}$					
4	$\frac{7553}{1305}$	$-\frac{8546}{3915}$	$\frac{9704}{1305}$	$-\frac{2272}{3915}$	$\frac{128}{145}$				
5	$\frac{29339076}{7553201}$	$-\frac{21443277}{7553201}$	$\frac{28916500}{7553201}$	$\frac{16930750}{7553201}$	$\frac{2376875}{7553201}$	$\frac{1250}{5399}$			
6	$\frac{30907943}{7553201}$	$-\frac{764212904}{245358795}$	$\frac{569814598}{49071759}$	$-\frac{24404100}{16357253}$	$\frac{40587816}{16357253}$	$-\frac{9716112}{81786265}$	$\frac{1728}{14723}$		
7	$\frac{1893308419143759}{341564845640503}$	$\frac{3055916255412910}{341564845640503}$	$\frac{2631114082150526}{341564845640503}$	$\frac{655352538774556}{341564845640503}$	$\frac{429893276819510}{341564845640503}$	$\frac{153686559046888}{341564845640503}$	$\frac{14020621454084}{341564845640503}$	$\frac{941192}{42817169}$	
8	$\frac{390240699042549}{53225681516269}$	$-\frac{1258409519494146}{372579770613883}$	$\frac{738033368179592}{53225681516269}$	$-\frac{655352538774556}{53225681516269}$	$\frac{206064363690624}{341564845640503}$	$\frac{15977115734016}{53225681516269}$	$\frac{17592680595456}{53225681516269}$	$-\frac{4236363300864}{372579770613883}$	$\frac{524288}{62721977}$

*Key: DP means degree of polynomial

Table A2: Coefficients of the Probabilists' Hermite Polynomial of Degree 8 for Example 2-5

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
Coefficients for Example 2								
$-\frac{6074914588167656833}{1150167065703461349}$	$\frac{6783616975860893500}{1150167065703461349}$	$-\frac{3719598990301129202}{1150167065703461349}$	$\frac{504211342112928016}{383389021901153783}$	$-\frac{145194492308783904}{383389021901153783}$	$\frac{30594587735513088}{383389021901153783}$	$-\frac{151423824388887424}{1150167065703461349}$	$-\frac{1637811908182016}{1150167065703461349}$	$-\frac{148207752839168}{1150167065703461349}$
Coefficients for Example 3								
$-\frac{8976000252997673335}{658990535398747026}$	$\frac{4707916202334829351}{219663511799582342}$	$-\frac{1045834716906648865}{658990535398747026}$	$\frac{7959863384878250960}{109831755899791171}$	$-\frac{244348962901420180}{109831755899791171}$	$\frac{51400632331030528}{109831755899791171}$	$-\frac{21897792528709600}{329495267699373513}$	$\frac{649894097092608}{109831755899791171}$	$-\frac{86616231116800}{329495267699373513}$
Coefficients for Example 4								
1.187403929476322	6.459728807308996	1.967257125642741	3.895306715052507	0.290706354863085	0.566821605947816	0.006149481021016	0.021673238258540	3.346751571139964e− 07
Coefficients for Example 5								
−2.026555661803312	−2.261644651969334	−5.442046244482299	−1.675341414084676	−1.725746194260807	−0.142297526484199	9.560777871221032e− 08	−0.117624085392121	−0.003549679778815

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