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Existence and Extremal Solution for a Fractional Order Differential Equation in Banach Algebras

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Abstract: *In this Paper studies the existence of solution for a fractional order nonlinear quadratic differential equation with initial value condition in banach algebras. Moreover; we show that solutions of this equation are locally attractive. We make use of the standard tools of the hybrid fixed point theory for two operators to establish the main result.. The existence theorems for extremal solutions are also proved under certain monotonicity conditions. Finally, our results are illustrated by a concrete example.*

Keywords: *Fractional Order Quadratic Differential Equation, Fixed Point Theorem, LocallyAttractivitty and Extremal Solutions, Banach Space.*

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I.Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium etc. involves derivatives of fractional order [1,5,10]. Recently, many authors have studied fractional Order differential equations from two aspects, one is the theoretical aspects of existence and uniqueness of solutions, the other is the analytic and numerical methods for finding solutions. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining more and more attention. For some recent development on the topic, see [3,7, 11] and the references therein. Nonlinear differential equation of fractional order plays an important role in branch of nonlinear analysis and their applications. There are different methods for dealing with the nonlinear differential equations.

As fixed point theory constitutes an important and core part of the subject of nonlinear analysis. The fixed point method is powerful technique that I have used for the existence the solution of fractional order nonlinear differential equation. This method has been shown to effectively, easily and accurately to solve a large class of nonlinear problems. At present there are several fixed point theorems which are useful in applications to nonlinear differential and integral equations. The selection of the fixed point theorem depends upon the given data.

We consider the following Fractional Order Nonlinear Quadratic Differential Equation (FNQDE) with Initial Conditions:

$$\left.\begin{array}{r} \mathfrak{D}^{\xi}\left[\frac{x(t)}{f(t,x(t))}\right]+\lambda\left[\frac{x(t)}{f(t,x(t))}\right]=g\big(t,x(t)\big), t\in\mathbb{R}_{+} \\ x(t_0)=x_0\in\mathbb{R} \\ f\big(t_0,x(t_0)\big)=f(t_0,x_0)\in\mathbb{R} \end{array}\right\} (1.1)$$

for $\lambda > 0 \in \mathbb{R}$, $\xi \in (0,1)$

Where,$f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} - \{0\}$ and $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a solution of Fractional Order Nonlinear Quadratic Differential Equation (1.1) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$such that:

(i) The function $t \to \left[\frac{x(t)}{f(t,x(t))}\right]$ is continuous for each$x \in \mathbb{R}$.

(ii) x satisfies (1.1)

II. Preliminaries

In this section we give the definitions, notation, hypothesis and preliminary tools, which will be used in the sequel.

Let $\mathbb{X} = C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous real valued function on $\mathbb{R}_+$ and Ω be a subset of $\mathbb{X}$. Let a mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ be an operator and consider the following operator equation in $\mathbb{X}$, namely,

$x(t) = (\mathbb{A}x)(t)$, for all $t \in \mathbb{R}_+$ (2.1)

Below we give some different characterization of the solutions for operator equation (2.1) on $\mathbb{R}_+$. We need the following definitions.

Definition 2.1 [8]: We say that solution of the equation (2.1) are locally attractive if there exists a closed ball $\overline{B_r(0)}$ in the space $\mathcal{AC}(\mathbb{R}_+,\mathbb{R})$and for some real number $r > 0$ such that for arbitrary solution $x = x(t)$ and $y = y(t)$ of equation (2.1) belonging to $\overline{B_r(0)} \cap \Omega$ we have that

$\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0$ (2.2)

Definition 2.2[6]: Let $\mathbb{X}$ be a Banach space. A mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ is called Lipschitz if there is a constant $\alpha > 0$ such that,$\|\mathbb{A}x - \mathbb{A}y\| \leq \alpha\|x - y\|$for all $x, y \in \mathbb{X}$. If$\alpha < 1$, then $\mathbb{A}$ is called a contraction on $\mathbb{X}$ with the contraction constantα.

Definition 2.3 [4]: An operator $\mathbb{Q}$ on a Banach space $\mathbb{X}$ into itself is called compact if for any bounded subset $S$of $\mathbb{X}$, $\mathbb{Q}(S)$ is relatively compact subset of $\mathbb{X}$. If $\mathbb{Q}$ is continuous and compact, then it is called completely continuous on $\mathbb{X}$.

Definition 2.4[6]:(Dugunji and Granas) Let$\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathbb{Q}: \mathbb{X} \to \mathbb{X}$, be an operator (in general nonlinear). Then $\mathbb{Q}$ is called

i. Compact if $\mathbb{Q}(X)$ is relatively compact subset of $\mathbb{X}$.
ii. Totally compact if $\mathbb{Q}(S)$ is totally bounded subset of $\mathbb{X}$ for any bounded subset $S$of $\mathbb{X}$.
iii. Completely continuous if it is continuous and totally bounded operator on $\mathbb{X}$.

It is clear that every compact operator is totally bounded but the converse need not be true.

We recall the basic definitions of fractional calculus which are useful in what follows.

Definition 2.5 [9]: The Riemann – Liouville fractional derivative of order $\xi > 0, n-1 < \xi < n, n \in \mathcal{N}$ with lower limit zero for a function f is defined as $\mathfrak{D}^\xi f(t) = \frac{1}{\Gamma(1-\xi)}\frac{d}{dt}\int_0^t \frac{f(s)}{(t-s)^\xi}ds \quad, t > 0$

Such that $\mathfrak{D}^{-\xi} f(t) = I^\xi f(t) = \frac{1}{\Gamma(\xi)}\int_0^t \frac{f(s)}{(t-s)^{1-\xi}}ds$respectively.

Definition 2.6[8]: The Riemann-Liouville fractional integral of order $\xi > 0, n-1 < \xi < n, n \in \mathcal{N}$ with lower limit zero for a function f is defined by the formula: $I^\xi f(t) = \frac{1}{\Gamma(\xi)}\int_0^t \frac{f(s)}{(t-s)^{1-\xi}}ds\,, \quad t > 0$

where $\Gamma(\xi)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ξ defined by $\mathfrak{D}^\xi = \frac{d^\xi}{dt^\xi} = \frac{d}{dt}\circ I^{1-\xi}$.

Theorem 2.1[6]: (Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in$\mathcal{C}(\mathbb{R}_+,\mathbb{R})$, then it has a convergent subsequence.

Theorem 2.2[6]: A metric space X is compact iff every sequence in X has a convergent subsequence.

We employ a new hybrid fixed pint theorem proved by Dhage [2] which is the main tool in the existence theorem of solutions of FNQDE.

Theorem 2.3[2]: Let S be a non-empty, bounded and closed-convex subset of the Banach space $\mathbb{X}$ and let $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: S \to \mathbb{X}$ are two operators satisfying:

a) $\mathbb{A}$ is Lipschitz with a lipschitz constant α,
b) $\mathbb{B}$ is completely continuous, and
c) $\mathbb{A}x\mathbb{B}x \in S$ for all $x \in S$, and
d) $\alpha M < 1$where$M = \|\mathbb{B}(S)\|: \sup\{\|\mathbb{B}x\|: x \in S\}$.Then the operator equation $\mathbb{A}x\mathbb{B}x = x$ has a solution in$S$.

Existence Theory:

We seek the solution of (2.1) in the space $\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ of continuous and real – valued function defined on $\mathbb{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ by,

$$\|x\| = sup\{|x(t)|: t \in \mathbb{R}_+\}, \ \ (xy)(t) = x(t)y(t), \ \ t \in \mathbb{R}_+$$

Clearly, $\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ becomes a Banach space with respect to the above norm and the multiplication in it.

Definition 3.1[6]: A mapping $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is Caratheodory if:

i) $t \to g(t,x)$ is measurable for each $x \in \mathbb{R}$ and
ii) $(x) \to g(t,x,)$is continuous almost everywhere for $t \in \mathbb{R}_+$.

Furthermore a Caratheodory function g is $\mathcal{L}^1$ –Caratheodory if:

iii) For each real number $r > 0$ there exists a function $\hbar_r \in \mathcal{L}^1(\mathbb{R}_+,\mathbb{R})$ such that$|g(t,x)| \leq \hbar_r(t) a.e.\ \ t \in \mathbb{R}_+$ for all $x \in \mathbb{R}$ with $|x|_r \leq r$.

Finally a caratheodory function g is $\mathcal{L}^1_{\mathbb{X}}$ –caratheodory if:

iv) There exists a function $\hbar \in \mathcal{L}^1(\mathbb{R}_+,\mathbb{R})$ such that $|g(t,x)| \leq \hbar(t), \quad a.e.\ \ t \in \mathbb{R}_+$ for all $x \in \mathbb{R}$

For convenience, the function $\hbar$ is referred to as a bound function for g.

Lemma 3.1: Suppose that $\xi \in (0,1)$ and the function f, g satisfying FNQDE (1.1). Then x is the solution of the FNQDE (1.1) if and only if it is the solution of integral equation

$x(t) = f\big(t,x(t)\big)\left\{\frac{x_0}{f(t_0,x_0)} - \frac{\lambda}{\Gamma(\xi)}\int_{t_0}^t \frac{x(s)}{f\big(s,x(s)\big)(t-s)^{1-\xi}}ds + \frac{1}{\Gamma(\xi)}\int_{t_0}^t \frac{g\big(s,x(s)\big)}{(t-s)^{1-\xi}}ds\right\}$(3.1) for all $t \in \mathbb{R}_+$

Proof: Integrating equation (1.1) of fractional order ξw.r.to, we get,

$$\mathfrak{D}^{\xi} I^{\xi}\left[\frac{x(t)}{f(t,x(t))}\right]+\lambda I^{\xi}\left[\frac{x(t)}{f(t,x(t))}\right]=I^{\xi}\left[g(t,x(t))\right]$$

$$\left[\frac{x(t)}{f(t,x(t))}\right]_{t_0}^{t}+\lambda I^{\xi}\left[\frac{x(t)}{f(t,x(t))}\right]=I^{\xi}\left[g(t,x(t))\right]$$

$$\frac{x(t)}{f(t,x(t))}-\frac{x(t_0)}{f(t_0,x(t_0))}+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds=\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}g(s,x(s))\,(t-s)^{\xi-1}ds$$

$$x(t)=f(t,x(t))\left\{\frac{x(t_0)}{f(t_0,x(t_0))}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s))}{(t-s)^{1-\xi}}ds\right\}$$

Since $x(t_0)=x_0\in\mathbb{R}$ and $f(t_0,x(t_0))=f(t_0,x_0)\in\mathbb{R}$

Conversely differentiate (3.1) of order ξw.r.to t, we get,

$$\mathfrak{D}^{\xi}\left[\frac{x(t)}{f(t,x(t))}\right]=\mathfrak{D}^{\xi}\left\{\frac{x(t_0)}{f(t_0,x(t_0))}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s))}{(t-s)^{1-\xi}}ds\right\}$$

$$\mathfrak{D}^{\xi}\left[\frac{x(t)}{f(t,x(t))}\right]=\mathfrak{D}^{\xi}\left[\frac{x(t_0)}{f(t_0,x(t_0))}\right]-\lambda\mathfrak{D}^{\xi}I^{\xi}\left[\frac{x(t)}{f(t,x(t))}\right]+\mathfrak{D}^{\xi}I^{\xi}\left[g(t,x(t))\right]$$

$$\mathfrak{D}^{\xi}\left[\frac{x(t)}{f(t,x(t))}\right]=0-\lambda\left[\frac{x(t)}{f(t,x(t))}\right]+\left[g(t,x(t))\right]$$

$$\mathfrak{D}^{\xi}\left[\frac{x(t)}{f(t,x(t))}\right]+\lambda\left[\frac{x(t)}{f(t,x(t))}\right]=\left[g(t,x(t))\right]$$

We need following hypothesis for existence the solution of fractional order nonlinear quadratic differential equation (FNQDE) (1.1).

($\mathcal{H}_1$) The function $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}-\{0\}$ is continuous and bounded with bound $\mathbb{F}=sup_{(t,x(t))\in\mathbb{R}_+\times\mathbb{R}}|f(t,x(t))|$ .There exist a bounded function $\alpha:\mathbb{R}_+\to\mathbb{R}$ with bound $\|\alpha\|$ satisfying: $|f(t,x(t))-f(t,y(t))|\le\alpha(t)|x(t)-y(t)|$ for all $x,y\in\mathbb{R}$.

($\mathcal{H}_2$) The function $g(t,x)=g:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is satisfying caratheodory condition with continuous function $h(t):\mathbb{R}_+\to\mathbb{R}$ such that $g(t,x)\le h(t)\forall t\in\mathbb{R}_+$ and $x\in\mathbb{R}$.

($\mathcal{H}_3$) The function $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}-\{0\}$ is satisfying caratheodory condition with continuous function $\mathbb{p}(t):\mathbb{R}_+\to\mathbb{R}$ such that $\frac{x(t)}{f(t,x(t))}\le\mathbb{p}(t),\forall t\in\mathbb{R}_+$ and $x\in\mathbb{R}$.

($\mathcal{H}_4$) The function $u,v:\mathbb{R}_+\to\mathbb{R}$ defined by the formulas $u(t)=\int_0^t\frac{\mathbb{p}(s)}{(t-s)^{1-\xi}}ds$ and $v(t)=\int_0^t\frac{h(s)}{(t-s)^{1-\xi}}ds$ is bounded on $\mathbb{R}_+$ and the functions $\mathbb{p}(t),u(t)andv(t)$ vanish at infinity.

Remark 3.1: Note that the $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$ hold, then there exists a constant $K_1,K_2>0$ such that $K_1=\sup\left\{\frac{\lambda u(t)}{\Gamma(\xi)}:t\in\mathbb{R}_+\right\}$ and $K_2=\sup\left\{\frac{v(t)}{\Gamma(\xi)}:t\in\mathbb{R}_+\right\}$ for all $t\in\mathbb{R}_+$ and $\mathbb{p}+K_1+K_2=\mathbb{K}$ say.

Main Result:

In this section we consider the FNQDE (1.1). The above hybrid fixed point theorem for three operators in Banach algebras $\mathbb{X}$, due to B.C.Dhage [2] will be used to prove existence the solution for given equation (1.1).

Theorem 4.1: Assume that conditions $(\mathcal{H}_1)-(\mathcal{H}_4)$ hold. Further if $\mathbb{FK}<r$ and$\mathbb{K}K_1<1$, where $\mathbb{K}$ and K_1 is defined in remark (3.1). Then FNQDE (1.1) has a solution in the space$\mathcal{C}(\mathbb{R}_+,\mathbb{R})$, moreover solution of (1.1) are locally attractive on $\mathbb{R}_+$.

Proof: By a solution of FNQDE (1.1) we mean a continuous function $\mathbb{X}:\mathbb{R}_+\to\mathbb{R}$ that satisfies FNQDE (1.1) on $\mathbb{R}_+$. Set $\mathbb{X}=\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ and define a subset S of$\mathbb{X}$as $S=\{x\in\mathbb{X}:\|x\|\le r\}$. Where$r$satisfies the inequality, $\mathbb{FK}\le r$.

Let $\mathbb{X}=\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ be a Banach Algebra of all continuous real-valued function on $\mathbb{R}_+$ with the norm,

$\|x\|=sup|x(t)|,t\in\mathbb{R}_+$ (4.1)

We shall obtain the solution of FNQDE (1.1) under some suitable conditions involved in (1.1)

Now the FNQDE (1.1) is equivalent to the FNQIE (3.1)

Let us define the two mappings $\mathbb{A}:\mathbb{X}\to\mathbb{X}and\mathbb{B}:S\to\mathbb{X}$ by,

$\mathbb{A}x(t)=f(t,x(t)),t\in\mathbb{R}_+$ (4.2)

$\mathbb{B}x(t)=\frac{x(t_0)}{f(t_0,x(t_0))}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s))}{(t-s)^{1-\xi}}ds,\ t\in\mathbb{R}_+$ (4.3)

Thus from the FNQDE (1.1) we obtain the operator equation as follows:

$x(t)=\mathbb{A}x(t)\mathbb{B}x(t),\ t\in\mathbb{R}_+$ (4.4)

If the operator $\mathbb{A}$ and $\mathbb{B}$ satisfy all the hypothesis of theorem (2.3), then the operator equation (4.4) has a solution on S.

Step I: Firstly we show that $\mathbb{A}$ is Lipschitz on S . Let $x, y \in \overline{B_r(0)}$; then by $(\mathcal{H}_1)$,

$$|\mathbb{A}x(t) - \mathbb{A}y(t)| \leq \left|f\big(t, x(t)\big) - f\big(t, y(t)\big)\right| \\ \leq \alpha(t)|x(t) - y(t)|$$

$\leq \alpha(t)|x(t) - y(t)|$for all $t \in \mathbb{R}_+$, $x, y \in S$

Taking suprimum over t we get,

$\|\mathbb{A}x - \mathbb{A}y\| \leq \|\alpha\|\|x - y\|$for all $x, y \in S$

Thus, $\mathbb{A}$ is Lipchitz on S with Lipschitz constant$\|\alpha\|$.

Step II: To show the operator$\mathbb{B}$ is completely continuous on$\mathbb{X}$. Let $\{x_n\}$ be a sequence in S converging to a point x . Then by lebesgue dominated convergence theorem for all $t \in \mathbb{R}_+$, we obtain $\lim_{n\to\infty} \mathbb{B}x_n(t)$

$$= \lim_{n\to\infty} \left\{\frac{x_n(t_0)}{f\big(t_0, x_n(t_0)\big)} - \frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t} \frac{x_n(s)}{f\big(s, x_n(s)\big)}(t-s)^{\xi-1}ds + \frac{1}{\Gamma(\xi)}\int_{t_0}^{t} \frac{g\big(s, x_n(s)\big)}{(t-s)^{1-\xi}}ds\right\}$$

$$= \frac{x(t_0)}{f\big(t_0, x(t_0)\big)} - \frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t} \frac{x(s)}{f\big(s, x(s)\big)}(t-s)^{\xi-1}ds + \frac{1}{\Gamma(\xi)}\int_{t_0}^{t} \frac{g\big(s, x(s)\big)}{(t-s)^{1-\xi}}ds$$

$$= \mathbb{B}x(t), \forall t \in \mathbb{R}_+$$

This shows that $\mathbb{B}$ is continuous on S.

Next we will prove that the set $\mathbb{B}(S)$ is uniformly bounded in$S$, for any$x \in S$, we have,

$$|\mathbb{B}x(t)| = \left|\frac{x(t_0)}{f\big(t_0, x(t_0)\big)} - \frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t} \frac{x(s)}{f\big(s, x(s)\big)}(t-s)^{\xi-1}ds + \frac{1}{\Gamma(\xi)}\int_{t_0}^{t} \frac{g\big(s, x(s)\big)}{(t-s)^{1-\xi}}ds\right|$$

$$\leq \left|\frac{x(t_0)}{f\big(t_0, x(t_0)\big)}\right| + \left|-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t} \frac{x(s)}{f\big(s, x(s)\big)}(t-s)^{\xi-1}ds\right| + \left|\frac{1}{\Gamma(\xi)}\int_{t_0}^{t} \frac{g\big(s, x(s)\big)}{(t-s)^{1-\xi}}ds\right|$$

$$\leq \mathbb{p}_0 + \frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t} \left|\frac{x(s)}{f\big(s, x(s)\big)}\right|(t-s)^{\xi-1}ds + \frac{1}{\Gamma(\xi)}\int_{t_0}^{t} \frac{\left|g\big(s, x(s)\big)\right|}{(t-s)^{1-\xi}}ds$$

$$\leq \mathbb{p}_0 + \frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t} \frac{\mathbb{p}(s)}{(t-s)^{1-\xi}}ds + \frac{1}{\Gamma(\xi)}\int_{t_0}^{t} \frac{h(s)}{(t-s)^{1-\xi}}ds$$

Taking supremum over t, we obtain

$\|\mathbb{B}x\| \leq \mathbb{p}_0 + \frac{\lambda u(t)}{\Gamma(\xi)} + \frac{v(t)}{\Gamma(\xi)} \leq \mathbb{p} + K_1 + K_2 = \mathbb{K}$say.

Therefore$\|\mathbb{B}x\| \leq \mathbb{K}$, which shows that $\mathbb{B}$ is uniformly bounded on S.

Now we will show that$\mathbb{B}$(S) is equicontinuous set in$\mathbb{X}$. Let $t_1, t_2 \in \mathbb{R}_+$ with $t_2 > t_1$and $x \in S$, then we have

$|\mathbb{B}x(t_2) - \mathbb{B}x(t_1)|$

$$= \left|\begin{array}{l} \left\{\frac{x(t_0)}{f\big(t_0, x(t_0)\big)} - \frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t_2} \frac{x(s)}{f\big(s, x(s)\big)}(t_2-s)^{\xi-1}ds + \frac{1}{\Gamma(\xi)}\int_{t_0}^{t_2} \frac{g\big(s, x(s)\big)}{(t_2-s)^{1-\xi}}ds\right\} \\ -\left\{\frac{x(t_0)}{f\big(t_0, x(t_0)\big)} - \frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t_1} \frac{x(s)}{f\big(s, x(s)\big)}(t_1-s)^{\xi-1}ds + \frac{1}{\Gamma(\xi)}\int_{t_0}^{t_1} \frac{g\big(s, x(s)\big)}{(t_1-s)^{1-\xi}}ds\right\} \end{array}\right|$$

$$\leq \left|\frac{x(t_0)}{f\big(t_0, x(t_0)\big)} - \frac{x(t_0)}{f\big(t_0, x(t_0)\big)}\right| + \left|\begin{array}{l} \frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t_2} \frac{x(s)}{f\big(s, x(s)\big)}(t_2-s)^{\xi-1}ds \\ -\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t_1} \frac{x(s)}{f\big(s, x(s)\big)}(t_1-s)^{\xi-1}ds \end{array}\right|$$

$$+\left|\frac{1}{\Gamma(\xi)}\int_{t_0}^{t_2} \frac{g\big(s,x(s)\big)}{(t_2-s)^{1-\xi}}ds - \frac{1}{\Gamma(\xi)}\int_{t_0}^{t_1} \frac{g\big(s,x(s)\big)}{(t_1-s)^{1-\xi}}ds\right|$$

$$\leq \frac{\lambda}{\Gamma(\xi)}\left|\int_{t_0}^{t_2} \mathbb{p}(s)(t_2-s)^{\xi-1}ds - \int_{t_0}^{t_1} \mathbb{p}(s)(t_1-s)^{\xi-1}ds\right| \\ + \frac{1}{\Gamma(\xi)}\left|\int_{t_0}^{t_2} h(s)(t_2-s)^{\xi-1}ds - \int_{t_0}^{t_1} h(s)(t_1-s)^{\xi-1}ds\right|$$

$$\leq \frac{\lambda}{\Gamma(\xi)}\|\mathbb{p}\|_{\mathcal{L}^1}\left|\begin{array}{l}\int_{t_0}^{t_2}(t_2-s)^{\xi-1}ds-\int_{t_0}^{t_2}(t_1-s)^{\xi-1}ds\\+\int_{t_0}^{t_2}(t_1-s)^{\xi-1}ds-\int_{t_0}^{t_1}(t_1-s)^{\xi-1}ds\end{array}\right|$$

$$+\frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left|\begin{array}{l}\int_{t_0}^{t_2}(t_2-s)^{\xi-1}ds-\int_{t_0}^{t_2}(t_1-s)^{\xi-1}ds\\+\int_{t_0}^{t_2}(t_1-s)^{\xi-1}ds-\int_{t_0}^{t_1}(t_1-s)^{\xi-1}ds\end{array}\right|$$

$$\leq \frac{\lambda}{\Gamma(\xi)}\|\mathbb{p}\|_{\mathcal{L}^1}\left\{\begin{array}{c}\left|\int_{t_0}^{t_2}\left[(t_2-s)^{\xi-1}-(t_1-s)^{\xi-1}\right]ds\right|\\+\left|\int_{t_1}^{t_2}(t_1-s)^{\xi-1}ds\right|\end{array}\right\}+\frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left\{\begin{array}{c}\left|\int_{t_0}^{t_2}\left[(t_2-s)^{\xi-1}-(t_1-s)^{\xi-1}\right]ds\right|\\+\left|\int_{t_1}^{t_2}(t_1-s)^{\xi-1}ds\right|\end{array}\right\}$$

$$\leq \frac{\lambda}{\Gamma(\xi)}\|\mathbb{p}\|_{\mathcal{L}^1}\left\{\left|\left[\frac{(t_2-s)^{\xi}}{-\xi}\right]_{t_0}^{t_2}-\left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_{t_0}^{t_2}\right|+\left|\left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_{t_1}^{t_2}\right|\right\}$$

$$+\frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left\{\left|\left[\frac{(t_2-s)^{\xi}}{-\xi}\right]_{t_0}^{t_2}-\left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_{t_0}^{t_2}\right|+\left|\left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_{t_1}^{t_2}\right|\right\}$$

$$\leq \frac{\lambda}{\Gamma(\xi+1)}\|\mathbb{p}\|_{\mathcal{L}^1}\left\{\begin{array}{c}\left|\begin{array}{c}-\left[(t_2-t_2)^{\xi}-(t_2-t_0)^{\xi}\right]+\\\left[(t_1-t_2)^{\xi}-(t_1-t_0)^{\xi}\right]\end{array}\right|+\\\left|-\left[(t_1-t_2)^{\xi}-(t_1-t_1)^{\xi}\right]\right|\end{array}\right\}+\frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi+1)}\left\{\begin{array}{c}\left|\begin{array}{c}-\left[(t_2-t_2)^{\xi}-(t_2-t_0)^{\xi}\right]+\\\left[(t_1-t_2)^{\xi}-(t_1-t_0)^{\xi}\right]\end{array}\right|+\\\left|-\left[(t_1-t_2)^{\xi}-(t_1-t_1)^{\xi}\right]\right|\end{array}\right\}$$

$$\leq\left\{\frac{\lambda}{\Gamma(\xi+1)}\|\mathbb{p}\|_{\mathcal{L}^1}+\frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi+1)}\right\}\left\{\left|(t_2-t_0)^{\xi}-(t_1-t_0)^{\xi}\right|\right\}$$

$\to 0$ as $t_1 \to t_2, \forall n \in \mathcal{N}$.

Implies $\mathbb{B}$ is equicontinuous.

Therefore by Arzela- Ascoli theorem that $\mathbb{B}$ is completely continuous operator on S.

Step III: To show $x = \mathbb{A}x\mathbb{B}y \Longrightarrow x \in S, \forall y \in S$

Let $x \in \mathbb{X}$, and $y \in S$ such that $x = \mathbb{A}x\mathbb{B}x$

By assumptions $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$

$$|x(t)| = |\mathbb{A}x(t)\mathbb{B}x(t)|$$
$$\leq |\mathbb{A}x(t)||\mathbb{B}x(t)|$$
$$\leq |f(t,x(t))|\left|\frac{x(t_0)}{f(t_0,x(t_0))}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s))}{(t-s)^{1-\xi}}ds\right|$$
$$\leq \mathbb{F}\left\{\left|\frac{x(t_0)}{f(t_0,x(t_0))}\right|+\left|-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds\right|+\left|\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s))}{(t-s)^{1-\xi}}ds\right|\right\}$$
$$\leq \mathbb{F}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\left|\frac{x(s)}{f(s,x(s))}\right|(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{|g(s,x(s))|}{(t-s)^{1-\xi}}ds\right\}$$
$$\leq \mathbb{F}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{\mathbb{p}(s)}{(t-s)^{1-\xi}}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{1-\xi}}ds\right\}$$

Taking supremum over t, we obtain

$$\leq \mathbb{F}\left\{\mathbb{p}_0+\frac{\lambda u(t)}{\Gamma(\xi)}+\frac{v(t)}{\Gamma(\xi)}\right\} \leq \mathbb{F}\{\mathbb{p}_0+K_1+K_2\} = \mathbb{F}\mathbb{K} \leq r$$

Therefore $\|x\| \leq \mathbb{F}\mathbb{K} \leq r$,

That is we have, $\|x\| = \|\mathbb{A}x\mathbb{B}x\| \leq r, \forall x \in S$.

Hence assumption (c) of theorem (2.3) is proved.

Step IV: Also we have

$$M = \|\mathbb{B}(S)\| = sup\{\|\mathbb{B}x\| : x \in S\}$$
$$= sup\left\{\begin{array}{c}sup_{t\in\mathbb{R}_+}\left[\left|\frac{x(t_0)}{f(t_0,x(t_0))}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s))}{(t-s)^{1-\xi}}ds\right|\right]\\: x \in S\end{array}\right\}$$

$$\leq sup\left\{sup_{t\in\mathbb{R}_+}\left\{\left|\frac{x(t_0)}{f(t_0,x(t_0))}\right|+\left|-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds\right|+\left|\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s))}{(t-s)^{1-\xi}}ds\right|\right\} : x\in S\right\}$$

$$\leq sup_{t\in\mathbb{R}_+}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\left|\frac{x(s)}{f(s,x(s))}\right|(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{|g(s,x(s))|}{(t-s)^{1-\xi}}ds\right\}$$

$$\leq sup_{t\in\mathbb{R}_+}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{\mathbb{p}(s)}{(t-s)^{1-\xi}}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{1-\xi}}ds\right\}$$

taking supremum over t, we obtain

$$\leq\left\{\mathbb{p}_0+\frac{\lambda u(t)}{\Gamma(\xi)}+\frac{v(t)}{\Gamma(\xi)}\right\}$$

$\leq\{\mathbb{p}_0+K_1+K_2\}=\mathbb{K}$

and therefore $MK=\mathbb{K}K<1$

Thus the condition (d) of theorem (2.3) is satisfied.

Hence all the conditions of theorem (2.3) are satisfied and therefore the operator equation $\mathbb{A}x\mathbb{B}x=x$ has a solution in **.** As a result, the FNQDE (1.1) has a solution defined on $\mathbb{R}_+$.

Step V: Finally we have to show that the locally attractivity of the solution for FNQDE (1.1). Let x and y be two solutions of FNQDE (1.1) in S defined on $\mathbb{R}_+$.

Then we have

$$|x(t)-y(t)|=\left|\begin{array}{l}\left\{[f(t,x(t))]\left[\frac{x(t_0)}{f(t_0,x(t_0))}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s))}{(t-s)^{1-\xi}}ds\right]\right\}-\\ \left\{[f(t,y(t))]\left[\frac{y(t_0)}{f(t_0,y(t_0))}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{y(s)}{f(s,y(s))}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,y(s))}{(t-s)^{1-\xi}}ds\right]\right\}\end{array}\right|$$

$$\leq\left\{|f(t,x(t))|\left[\left|\frac{x(t_0)}{f(t_0,x(t_0))}\right|+\left|-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f(s,x(s))}(t-s)^{\xi-1}ds\right|+\left|\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,x(s))}{(t-s)^{1-\xi}}ds\right|\right]\right\}+$$
$$\left\{|f(t,y(t))|\left[\left|\frac{y(t_0)}{f(t_0,y(t_0))}\right|+\left|-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{y(s)}{f(s,y(s))}(t-s)^{\xi-1}ds\right|+\left|\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g(s,y(s))}{(t-s)^{1-\xi}}ds\right|\right]\right\}$$

$$\leq\mathbb{F}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\left|\frac{x(s)}{f(s,x(s))}\right|(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{|g(s,x(s))|}{(t-s)^{1-\xi}}ds\right\}+$$
$$\mathbb{F}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\left|\frac{y(s)}{f(s,y(s))}\right|(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{|g(s,y(s))|}{(t-s)^{1-\xi}}ds\right\}$$

$$\leq\mathbb{F}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{\mathbb{p}(s)}{(t-s)^{1-\xi}}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{1-\xi}}ds\right\}+$$
$$\mathbb{F}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{\mathbb{p}(s)}{(t-s)^{1-\xi}}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{1-\xi}}ds\right\}$$

$$\leq 2\mathbb{F}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{\mathbb{p}(s)}{(t-s)^{1-\xi}}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{1-\xi}}ds\right\}$$

Taking supremum over t, we obtain

$$\leq\mathbb{F}\left\{\mathbb{p}_0+\frac{\lambda u(t)}{\Gamma(\xi)}+\frac{v(t)}{\Gamma(\xi)}\right\}$$

Since $\lim_{t\to\infty}v(t)=0$, $\lim_{t\to\infty}u(t)=0$, $\lim_{t\to\infty}\mathbb{p}(t)=0$ for $\epsilon>0$, there exist a real number $\mathbb{T}'>0,\mathbb{T}''>0$ and $\mathbb{T}'''>0$ such that $\mathbb{p}_0\leq\frac{\epsilon}{6\mathbb{F}}$, $u(t)\leq\frac{\Gamma(\xi)\epsilon}{\lambda 6\mathbb{F}}$ and $v(t)\leq\frac{\Gamma(\xi)\epsilon}{6\mathbb{F}}$ for all $t\geq\mathbb{T}^*$, if we choose $\mathbb{T}^*=max\{\mathbb{T}',\mathbb{T}'',\mathbb{T}'''\}$.

Then from above inequality it follows that $|x(t)-y(t)|<\epsilon$ for all $t\geq\mathbb{T}^*$.

Hence FNQIE (1.1) has a locally attractive solution on $\mathbb{R}_+$.

This completes the proof.

Existence of extremal solutions:

A closed and non-empty set $\mathbb{K}$ in a Banach Algebra $\mathbb{X}$ is called a cone if

i. $\mathbb{K}+\mathbb{K}\subseteq\mathbb{K}$

ii. $\lambda\mathbb{K}\subseteq\mathbb{K}$ for $\lambda\in\mathbb{R},\lambda\geq 0$

iii. $\{-\mathbb{K}\}\cap\mathbb{K}=0$ where 0 is the zero element of $\mathbb{X}$.

and is called positive cone if

iv. $\mathbb{K} \circ \mathbb{K} \subseteq \mathbb{K}$

And the notation $\circ$ is a multiplication composition in $\mathbb{X}$

We introduce an order relation $\leq$ in $\mathbb{X}$ as follows.

Let $x, y \in \mathbb{X}$ then $x \leq y$ if and only if $y - x \in \mathbb{K}$. A cone $\mathbb{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on$\mathbb{K}$. It is known that if the cone $\mathbb{K}$ is normal in $\mathbb{X}$ then every order-bounded set in $\mathbb{X}$ is norm-bounded set in $\mathbb{X}$.

We equip the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ of continuous real valued function on $\mathbb{R}_+$ with the order relation $\leq$ with the help of cone defined by,

$\mathbb{K} = \{x \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}): x(t) \geq 0 \ \forall t \in \mathbb{R}_+\}$ (5.1)

We well known that the cone $\mathbb{K}$ is normal and positive in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$. As a result of positivity of the cone $\mathbb{K}$ we have:

Lemma 5.1[3]: Let $p_1, p_2, q_1, q_2 \in \mathbb{K}$ be such that $p_1 \leq q_1$and $p_2 \leq q_2$then $p_1 p_2 \leq q_1 q_2$.

For any $p, q \in \mathbb{X} = \mathcal{C}(\mathbb{R}_+, \mathbb{R}), p \leq q$ the order interval $[p, q]$ is a set in $\mathbb{X}$ given by,

$[p, q] = \{x \in \mathbb{X}: p \leq x \leq q\}$(5.2)

Definition 5.1[3]**:** A mapping $G: [p, q] \to \mathbb{X}$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Gx \leq Gy$ for all $x, y \in [p, q]$.

For proving the existence of extremal solutions of the equations (1.1) under certain monotonicity conditions by using following fixed pint theorem of Dhage [3].

Theorem 5.1 [3] : Let $\mathbb{K}$ be a cone in Banach Algebra $\mathbb{X}$ and let $[p, q] \in \mathbb{X}$. Suppose that $\mathbb{A}, \mathbb{B}: [p, q] \to \mathbb{K}$ are two operators such that

a. $\mathbb{A}$ is a Lipschitz with Lipschitz constant α,

b. $\mathbb{B}$ is completely continuous,

c. $\mathbb{A}x\mathbb{B}x \in [p, q]$ for each $x \in [p, q]$ and

d. $\mathbb{A}$and$\mathbb{B}$ are nondecreasing.

Further if the cone $\mathbb{K}$ is normal and positive then the operator equation $\mathbb{A}x\mathbb{B}x = x$ has the least and greatest positive solution in $[p, q]$ whenever $\alpha M < 1$, where $M = \|\mathbb{B}([p, q])\| = sup\{\|\mathbb{B}x\|: x \in [p, q]\}$

We need following definitions and hypothesis for existence the extremal solution of FNQDE (1.1).

Definition 5.2: A function $p \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ is called a **lower solution** of the FNQDE (1.1) on $\mathbb{R}_+$ if the function $t \to \frac{p(t)}{f(t, p(t))}$ is continuous and

$$\left.\begin{array}{c}\mathfrak{D}^{\xi}\left[\dfrac{p(t)}{f(t, p(t))}\right] \leq g(t, p(t)) - \lambda\left[\dfrac{p(t)}{f(t, p(t))}\right], a.e., t \in \mathbb{R}_+ \\ p(t_0) = p_0 \\ f(t_0, p_0) = f(t_0, p_0)\end{array}\right\}$$

Again a function $q \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ is called an **upper solution** of the FNQDE (1.1) on $\mathbb{R}_+$ if function the $t \to \frac{q(t)}{f(t, q(t))}$ is continuous and

$$\left.\begin{array}{c}\mathfrak{D}^{\xi}\left[\dfrac{q(t)}{f(t, q(t))}\right] \geq g(t, q(t)) - -\lambda\left[\dfrac{q(t)}{f(t, q(t))}\right], a.e., t \in \mathbb{R}_+ \\ q(t_0) = q_0 \\ f(t_0, q_0) = f(t_0, q_0)\end{array}\right\}$$

Definition 5.3: A solution x_M of the FNQDE (1.1) is said to be maximal if for any other solution x to FNQDE (1.1) one has $x(t) \leq x_M(t)$ for all t $\in \mathbb{R}_+$. Again a solution x_M of the FNQDE (1.1) is said to be minimal if $x_M(t) \leq x(t)$ for all t $\in \mathbb{R}_+$ where x is any solution of the FNQDE (1.1) on $\mathbb{R}_+$.

Definition 5.4 (Caratheodory Case):

We consider the following set of assumptions:

$\mathfrak{B}1)$ The functions $g(t, x(t))$ and $\frac{x(t)}{f(t, x(t))}$ are Caratheodory.

$\mathfrak{B}2)$ The functions $(t, x(t))$, $g(t, x(t))$ and $\frac{x(t)}{f(t, x(t))}$ are non-decreasing in x almost everywhere for all $t \in \mathbb{R}_+$.

$\mathfrak{B}3)$ The FNQDE (1.1) has a lower solution p and an upper solution q on $\mathbb{R}_+$with $p \leq q$.

$\mathfrak{B}4)$ The function $l: \mathbb{R}_+, \mathbb{R}$ defined by,$l(t) = |g(t, p(t))| + |g(t, q(t))|$is Lebesgue measurable.

$\mathfrak{B}5)$ The function $\mathbb{q}: \mathbb{R}_+, \mathbb{R}$ defined by,$\mathbb{q}(t) = \left|\frac{p(t)}{f(t, p(t))}\right| + \left|\frac{q(t)}{f(t, q(t))}\right|$is Lebesgue measurable.

Remark 5.1: Assume that $(\mathfrak{B}1 - \mathfrak{B}5)$ hold. Then$|g(t, x(t))| \leq l(t), a.e. t \in \mathbb{R}_+$, for all $x \in [p, q]$ and$\left|\frac{x(t)}{f(t, x(t))}\right| \leq \mathbb{q}(t)$.

Theorem 5.1: Suppose that the assumptions $(\mathcal{H}_1)-(\mathcal{H}_4)$ and $(\mathfrak{B}1)-(\mathfrak{B}5)$ holds and $l,\mathbb{q}$ are given in remark (5.1) and $\alpha\left\{\mathbb{p}_0+\frac{T^{\xi}}{\Gamma(\xi+1)}[\lambda\|\mathbb{q}\|_{\mathcal{L}^1}+\|l\|_{\mathcal{L}^1}]\right\}<1$ hold then FNQDE (1.1) has a minimal and maximal positive solution on $\mathbb{R}_+$.

Proof: Now FNQDE (1.1) is equivalent to FNQIE (3.1)on$\mathbb{R}_+$. Let $\mathbb{X}=\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ and define an order relation "$\leq$" by the cone $\mathbb{K}$ given by (5.1). Clearly $\mathbb{K}$ is a normal cone in $\mathbb{X}$. Define two operators $\mathbb{A}$ and $\mathbb{B}$ on $\mathbb{X}$ by (4.2) and (4.3) respectively. Then FNQIE (3.1) is transformed into an operator equation $\mathbb{A}x\mathbb{B}x=x$ in BanachAlgebra$\mathbb{X}$. Notice that $(\mathfrak{B}1)$ implies $\mathbb{A},\mathbb{B}:[p,q]\to\mathbb{K}$ Since the cone $\mathbb{K}$ in $\mathbb{X}$ is normal, $[p,q]$is a norm bounded set in $\mathbb{X}$. Now it is shown, as in the proof of Theorem (4.1), that $\mathbb{A}$ is a Lipschitz with a Lipschitz constant $\|\alpha\|$ and $\mathbb{B}$ is completely continuous operator on$[p,q]$.

Step I: Again the hypothesis $(\mathfrak{B}2)$ implies that $\mathbb{A}$ and $\mathbb{B}$ are non-decreasing on $[p,q]$. To see this, let $x,y\in[p,q]$ be such that $x\leq y$.then by $(\mathfrak{B}2)$ we have,

$\mathbb{A}x(t)=f\big(t,x(t)\big)\leq f\big(t,y(t)\big)\leq\mathbb{A}y(t),\forall t\in\mathbb{R}_+$

Similarly,

$$\begin{aligned}\mathbb{B}x(t)&=\frac{x(t_0)}{f\big(t_0,x(t_0)\big)}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f\big(s,x(s)\big)}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g\big(s,x(s)\big)}{(t-s)^{1-\xi}}ds\\&\leq\frac{y(t_0)}{f\big(t_0,y(t_0)\big)}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{y(s)}{f\big(s,y(s)\big)}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g\big(s,y(s)\big)}{(t-s)^{1-\xi}}ds\\&\leq\mathbb{B}y(t),\forall t\in\mathbb{R}_+\end{aligned}$$

Implies that $\mathbb{A}$ and $\mathbb{B}$ are non-decreasing operators on $[p,q]$.

Step II: Again definition (5.2) and hypothesis $(\mathfrak{B}3)$ implies that ,

$$\begin{aligned}p(t)&\leq f\big(t,p(t)\big)\left\{\frac{p(t_0)}{f\big(t_0,p(t_0)\big)}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{p(s)}{f\big(s,p(s)\big)}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g\big(s,p(s)\big)}{(t-s)^{1-\xi}}ds\right\}\\&\leq f\big(t,x(t)\big)\left\{\frac{x(t_0)}{f\big(t_0,x(t_0)\big)}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f\big(s,x(s)\big)}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g\big(s,x(s)\big)}{(t-s)^{1-\xi}}ds\right\}\\&\leq f\big(t,q(t)\big)\left\{\frac{q(t_0)}{f\big(t_0,q(t_0)\big)}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{q(s)}{f\big(s,q(s)\big)}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g\big(s,q(s)\big)}{(t-s)^{1-\xi}}ds\right\}\end{aligned}$$

$\leq q(t),\forall t\in\mathbb{R}_+andx\in[p,q]$

As a result $p(t)\leq\mathbb{A}x(t)\mathbb{B}x(t)\leq q(t),\forall t\in\mathbb{R}_+$ and $x\in[p,q]$

Hence $\mathbb{A}x\mathbb{B}x\in[p,q],\forall x\in[p,q]$

Step III: Again

$M=\|\mathbb{B}([p,q])\|=sup\{\|\mathbb{B}x\|:x\in[p,q]\}$

$$\begin{aligned}&\leq sup\left\{sup_{t\in\mathbb{R}_+}\left\{\left|\frac{x(t_0)}{f\big(t_0,x(t_0)\big)}-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f\big(s,x(s)\big)}(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g\big(s,x(s)\big)}{(t-s)^{1-\xi}}ds\right|\right\}\right\}\\&\leq sup\left\{\begin{matrix}sup_{t\in\mathbb{R}_+}\left\{\left|\frac{x(t_0)}{f\big(t_0,x(t_0)\big)}\right|+\left|-\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{x(s)}{f\big(s,x(s)\big)}(t-s)^{\xi-1}ds\right|+\left|\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{g\big(s,x(s)\big)}{(t-s)^{1-\xi}}ds\right|\right\}\\:x\in S\end{matrix}\right\}\\&\leq sup_{t\in\mathbb{R}_+}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\left|\frac{x(s)}{f\big(s,x(s)\big)}\right|(t-s)^{\xi-1}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{\big|g\big(s,x(s)\big)\big|}{(t-s)^{1-\xi}}ds\right\}\\&\leq sup_{t\in\mathbb{R}_+}\left\{\mathbb{p}_0+\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{\mathbb{q}(s)}{(t-s)^{1-\xi}}ds+\frac{1}{\Gamma(\xi)}\int_{t_0}^{t}\frac{l(s)}{(t-s)^{1-\xi}}ds\right\}\\&\leq\mathbb{p}_0+\frac{\lambda\|\mathbb{q}\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left|\left[\frac{(t-s)^{\xi}}{-\xi}\right]_{t_0}^{t}\right|+\frac{\|l\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left|\left[\frac{(t-s)^{\xi}}{-\xi}\right]_{t_0}^{t}\right|\\&\leq\mathbb{p}_0+\frac{\lambda\|\mathbb{q}\|_{\mathcal{L}^1}}{\Gamma(\xi+1)}\left\{\left|(t-t)^{\xi}-(t-t_0)^{\xi}\right|\right\}+\frac{\|l\|_{\mathcal{L}^1}}{\Gamma(\xi+1)}\left\{\left|(t-t)^{\xi}-(t-t_0)^{\xi}\right|\right\}\\&\leq\mathbb{p}_0+\frac{\lambda\|\mathbb{q}\|_{\mathcal{L}^1}}{\Gamma(\xi+1)}(t-t_0)^{\xi}+\frac{\|l\|_{\mathcal{L}^1}}{\Gamma(\xi+1)}(t-t_0)^{\xi}\\&\leq\mathbb{p}_0+\frac{\lambda\|\mathbb{q}\|_{\mathcal{L}^1}}{\Gamma(\xi+1)}T^{\xi}+\frac{\|l\|_{\mathcal{L}^1}}{\Gamma(\xi+1)}T^{\xi}\\&\leq\mathbb{p}_0+\frac{T^{\xi}}{\Gamma(\xi+1)}[\lambda\|\mathbb{q}\|_{\mathcal{L}^1}+\|l\|_{\mathcal{L}^1}]\end{aligned}$$

Since $\alpha M < 1$ that is $\alpha\left\{\mathbb{p}_0 + \frac{T^{\xi}}{\Gamma(\xi+1)}[\lambda\|\mathbb{q}\|_{\mathcal{L}^1} + \|l\|_{\mathcal{L}^1}]\right\} < 1$

We apply theorem (5.1) to the operator equation $\mathbb{A}x\mathbb{B}x = x$ to yield that the FNQDE (1.1) has minimum and maximum positive solution on $\mathbb{R}_+$.

This completes the proof.

Example:

Example: Consider the following FNQDE of type (1.1)

$$\mathfrak{D}^{\frac{1}{2}}\left[\frac{x(t)}{f(t,x(t))}\right] + 3\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), t \in \mathbb{R}_+$$

$$x(0) = 0 \qquad (6.1)$$

$$f(0,0) = f(0,0)$$

Where the functions

$f(t,x(t)) = sin4t\left[\frac{x(t)}{1-x(t)}\right]$, $g(t,x(t)) = \frac{1}{t(1+x(t))}$, $h(t) = \frac{1}{t}$ and $\mathbb{p}(t) = \frac{1}{sin\,4t}$

and $\xi = \frac{1}{2}$, $\lambda = 3$.

$(\mathcal{H}_1)$ Now $|f(t,x(t)) - f(t,y(t))|$

$$= \left|\left\{sin4t\left[\frac{x(t)}{1-x(t)}\right]\right\} - \left\{sin4t\left[\frac{y(t)}{1-y(t)}\right]\right\}\right|$$

$$= \left|sin4t\left[\frac{x(t)}{1-x(t)} - \frac{y(t)}{1-y(t)}\right]\right|$$

$$\leq |sin4t|\left|\frac{x(t)y(t) + x(t) - y(t) - x(t)y(t)}{x(t)y(t) - x(t) - y(t) + 1}\right|$$

$$\leq |sint||x(t) - y(t)|$$

$$\leq \alpha(t)|x(t) - y(t)|$$

$$\leq \|\alpha\||x(t) - y(t)|$$

Since $\alpha(t) = sin4t$ say which is continuous and bounded on $\mathbb{R}_+$ has bound $\|\alpha\|$.

$(\mathcal{H}_2)$Take $h(t) = \frac{1}{t}$, it is continuous on $\mathbb{R}_+$.

Implies $g(t,x(t)) \leq h(t)$ that is $\frac{1}{t(1+x(t))} \leq \frac{1}{t}$

Implies g is caratheodory satisfy above condition.

$(\mathcal{H}_3)$ The function $\frac{x(t)}{f(t,x(t))}$ is again caratheodory function with continuous function $\mathbb{p}(t): \mathbb{R}_+ \to \mathbb{R}$ such that $\mathbb{p}(t) = \frac{1}{sin\,4t}$ and satisfying $\frac{x(t)}{f(t,x(t))} \leq \mathbb{p}(t)$

It follows that all the conditions $(\mathcal{H}_1) - (\mathcal{H}_3)$ satisfied.

Thus by theorem (2.3) above problem has a solution on $\mathbb{R}_+$.

III. Conclusions

In this paper we have studied the existence of solution of fractional order nonlinear quadratic differential equation. The result has been obtained by using hybrid fixed point theorem for two operators in Banach space due to Dhage. The main result is well illustrated with the help of example.

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