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Robertson-Walker Model in Flat Space

MishaBillah, Lipon Chandra Das
Department of Mathematics, University of Chittagong, Bangladesh
Department of Mathematics, University of Chittagong, Bangladesh

Abstract: *In cosmology, one is supposed to derive the properties of early universe by solving FRW metric. According to the FRW model, the universe has a space-time singularity at a finite time in the past. This space time singularity is called the big-bang. Though the big-bang singularity can not describe what the conditions were at the very beginning of the universe, it can help scientists describe the earliest moments after the start of the expansion. In this paper an attempt has been made to describe the relation of radius and density with time of early universe and present universe by calculating different values of christoffels symbols, Ricci tensor , Ricci scalar and Einestein's equation for the flat space.*

***Keywords**: FRW model, Christoffel symbol, Ricci tensor, Ricci scalar, Energy momentum tensor, Einestein equation, Decleration parameter.*

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I. Introduction

Standard cosmology is based on the Friedmann Robertson Walker (FRW) metric for a spatially homogeneous and isotropic three-dimensional space. In terms of the proper time t measured by a co-moving observer and the corresponding radial (r) and angular (θ & φ) co-ordinates in the co-moving frame, the interval[1],

$$ds^2 = g_{\mu\upsilon}\, dx^\mu\, dx^\upsilon$$

Where $g_{\mu\upsilon}$ $(\mu, \upsilon = 0,1,2,3)$ are the metric co-efficients.

R-W model can be written as

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

The expansion factor $R(t)$ is a function of cosmic time , where the spatial co-ordinates (r, θ, φ) in this frame remain fixed for all particles in the cosmos. In this model, there are three kinds of space are flat, positive and negative constant curvature, which incorporate the closed and open models respectively. The model determines here constant $k = 0$ for flat space, +1 for a closed universe and -1 for an open universe. For flat space, R-W model can be written as[2]

$$ds^2 = dt^2 - R^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)] \;\cdots\cdots\cdots (1)$$

Here co-ordinate t is time like and other co-ordinates r, θ, φ are space like. θ & φ are the corresponding angular co-ordinates in the co-moving frame.

Robertson-Walker model also describes the initial moment of early universe that means big-bang. During the time of big-bang, the radius of the universe was very small. On the other hand the density and temperature of the universe was very high. Over time the radius of the universe continues to expand. At the same time the temperature and density were also decreasing. Here we try to prove it by calculating different values of christoffels symbols for the flat space[3].

II. Calculations

Now$x^\mu = (x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$

and the non-zero metric components and christoffel symbols are given below:

$$g_{00} = 1, \quad g_{11} = -R^2, \quad g_{22} = -R^2 r^2, \quad g_{33} = -R^2 r^2 \sin^2\theta$$

And we know

$$\Gamma^\mu_{\upsilon\lambda} = g^{\mu\rho}\left(\frac{1}{2} g_{\sigma\upsilon,\lambda} + g_{\sigma\lambda,\upsilon} - g_{\upsilon\lambda,\sigma}\right)$$

So, $$\Gamma^0_{11} = R\dot{R}, \quad \Gamma^0_{22} = R\dot{R}r^2, \quad \Gamma^0_{33} = R\dot{R}r^2\sin^2\theta,$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{R}}{R}, \quad \Gamma^1_{11} = 0 \quad \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{R}}{R},$$

$$\Gamma^1_{11} = 0, \Gamma^1_{22} = -r, \Gamma^1_{33} = -r\sin^2\theta,$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r},$$

$$\Gamma_{33}^{2} = -\sin\theta\cos\theta\,, \qquad \Gamma_{23}^{3} = \Gamma_{32}^{3} = \cot\theta,$$

Now we know Ricci tensor is

$$R_{jk} = \Gamma_{rk}^{i}\Gamma_{ji}^{r} - \Gamma_{ri}^{i}\Gamma_{jk}^{r} + \frac{\partial}{\partial x^{k}}\Gamma_{ji}^{i} - \frac{\partial}{\partial x^{i}}\Gamma_{jk}^{i}$$

$$\text{So, } R_{00} = \Gamma_{r0}^{i}\Gamma_{0i}^{r} - \Gamma_{ri}^{i}\Gamma_{00}^{r} + \frac{\partial}{\partial x^{0}}\Gamma_{0i}^{i} - \frac{\partial}{\partial x^{i}}\Gamma_{00}^{i}$$

$$\text{Now,} \Gamma_{r0}^{i}\Gamma_{0i}^{r} = \frac{3\dot{R}^{2}}{R^{2}}$$

$$\text{and} \Gamma_{ri}^{i}\Gamma_{00}^{r} = 0$$

$$\frac{\partial}{\partial x^{i}}\Gamma_{0i}^{i} = \frac{\partial}{\partial t}\left(3\frac{\dot{R}}{R}\right) = \frac{3(R\ddot{R} - \dot{R}^{2})}{R^{2}}$$

Thus we get,

$$R_{00} = \frac{3\ddot{R}}{R}$$

Now similarly we get,

$$R_{11} = -(R\ddot{R} + 2\dot{R}^{2})$$
$$R_{22} = -r^{2}(R\ddot{R} + 2\dot{R}^{2})$$
$$R_{33} = -r^{2}\sin^{2}\theta\,(R\ddot{R} + 2\dot{R}^{2})$$

Now Ricci scalar R is given by

$$R = R_{\mu\rho}R^{\mu\rho} = 6\frac{(R\ddot{R} + \dot{R}^{2})}{R^{2}}$$

Now we know from the Einestein field equation

$R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R = -8\pi G T_{\mu\upsilon}$ ………(2)

Where$T_{\mu\upsilon}$, the energy momentum tensor is given by

$$T_{\mu\upsilon} = (p+\rho)u_{\mu}u_{\upsilon} - p g_{\mu\upsilon}, \quad \text{where } \upsilon,\mu = 0,1,2,3 \cdots \cdots$$

So we get,

$$T_{00} = \rho$$
$$T_{11} = -p g_{11} = pR^{2}$$
$$T_{22} = -p g_{22} = pR^{2}r^{2}$$
$$\text{and} T_{33} = -p g_{33} = pR^{2}r^{2}\sin^{2}\theta$$

Now, $R_{00} - \frac{1}{2}g_{00}R = -8\pi G T_{00}$

or, $3H^{2} = 8\pi G\rho$

or,$(\frac{\dot{R}}{R})^{2} = \frac{8\pi G\rho}{3}$ ………(3)

This is the solution for Einestein's equation for zero-zero space.

Equation(3) can be written as

$$\frac{\rho}{\frac{3H^{2}}{8\pi G}} = 1$$

$$\text{or, } \frac{\rho}{\rho_{c}} = 1 \qquad as\ \rho_{c} = \frac{3H^{2}}{8\pi G}$$

$$\text{so} \Omega = 1$$

Now, $T^{\mu\upsilon}_{;\upsilon} = 0$

or, $T^{\mu\upsilon}_{,\upsilon} + \Gamma^{\mu}_{\lambda\upsilon}T^{\upsilon\lambda} + \Gamma^{\upsilon}_{\lambda\upsilon}T^{\mu\lambda} = 0$

or, $\dot{\rho} + 3H(p+\rho) = 0$ $\cdots\cdots\cdots(4)$

Which is called the conservation of energy equation.

When $p = 0$,equation(4) becomes

$$\frac{d\rho}{dt} + 3H\rho = 0$$

$$\text{or, } \frac{d\rho}{\rho} + 3\ \frac{dR}{R} = 0$$

$$\text{or, } ln\rho + 3lnR = c_{1} \quad [by\ integrating]$$

$$\text{or, } \ln(\rho R^{3}) = c_{1}$$

so we can write,

$$\rho\alpha\frac{1}{R^{3}}. \quad \text{………(5)} \quad \text{for matter dominant universe}$$

Again for $p = \frac{1}{3}\rho$,equation(4) becomes

$$\rho \,\alpha\, \frac{1}{R^4}. \quad \dots \dots \dots (6) \quad \text{for radiation dominant universe}$$

$$\text{Now,} \quad R_{11} - \frac{1}{2} g_{11} R = -8\pi G T_{11}$$

$$\text{or,} 2R\ddot{R} + \dot{R}^2 = -8\pi G p R^2$$

$$\text{or, } 2\frac{\ddot{R}}{R} + H^2 = -8\pi G p$$

$$\text{or, } 2\frac{\ddot{R}}{R} = -8\pi G p - \frac{8\pi G\rho}{3}$$

$$\text{so, } \ddot{R} = -\frac{4\pi G}{3}(3p + \rho)R, \ \text{ for space-space component}$$

When p=0, then

$$\ddot{R} = -\frac{4\pi G}{3}\rho(t)R(t)$$

$$\text{So, } \ddot{R} = -\frac{GM}{R^2}, \ \text{ as } \rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$\text{or, } \frac{1}{2} 2\dot{R}\ddot{R} = -\dot{R} G M R^{-2}$$

$$\text{or, } d\left(\frac{1}{2}\dot{R}^2\right) = d\left(\frac{GM}{R}\right)$$

By integrating, we get

$$\frac{1}{2}\dot{R}^2 = \frac{GM}{R} + E$$

Where E is a constant.
Now the declaration parameter is,

$$q = -\frac{\ddot{R}R}{\dot{R}^2}$$

$$= \frac{GM}{R}\frac{1}{\dot{R}^2}$$

$$= \left(\frac{1}{2}\dot{R}^2 - E\right)\frac{1}{\dot{R}^2}$$

$$\therefore q = \frac{1}{2} - \frac{E}{\dot{R}^2}$$

From this relation, we get

$$\text{If } E < 0, \ \text{ then } q > \frac{1}{2}$$

$$\text{If } E = 0, \ \text{ then } q = \frac{1}{2}$$

$$\text{If } E > 0, \ \text{ then } q < \frac{1}{2}$$

III. Conclusion

Now from the above equation, we can conclude the discussion by saying that the radius of the present universe is greater than that of early universe and the density of the present universe is less than that of early universe.Besides, energy of the early universe is also greater than the energy of present universe. Here it is also showed that, our universe is expanding day by day. So the radius of our universe is also increasing but the density is decreasing day by day.We can say it in other way that our universe is expanding.

References

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