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Representation of Soft Substructures of a Soft Group

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Abstract: *Our aim in this paper is to study some lattice theoretic properties of factorizable substructures of a product group which will play an important role in the representation of soft substructures of a soft group by certain crisp subgroups. In fact, for any soft group over a group, we construct a crisp group in such way that the complete lattice of all soft substructures of the former is complete epimorphic to a complete lattice of certain crisp substructures of the later.*

Keywords: *Associated product group for a soft group, (Extended) Soft (normal) subgroup, Factorizable (normal) subgroup, (Product) Group.*

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I. Introduction

Ever since, Molodtsov[17] introduced the new concept of a soft set over a universal set as a completely generic mathematical tool for modeling uncertainties, mathematicians started imposing and studying algebraic, topological and topologically algebraic structures on them. For a study on soft algebraic structures, one can refer to Aktas-Cagman[2] for soft groups; Sezgin-Atagun[24] for soft groups and normalistic soft groups; Feng-Jun-Zhao[12] for soft semi rings; Acar-Koyuncu-Tanay[1] for soft rings; Sun-Zhang-Liu[15] for soft modules; Sezgin-Atagun-Ayugn[25] for soft near-rings and idealistic soft near-rings; Atagun-Sezgin[6] for soft substructures of rings, fields and modules; Ali-Shabir-Shum[5] and Feng-Ali-Shabir[13] for soft semigroups and soft homomorphisms, Murthy-Maheswari[19] for *f(p)*-soft τ -algebras and their ω -subalgebras and Changphas-Thon-gkam[8] for soft algebras in a general viewpoint. For studies in soft topological spaces, one can refer to Shabir-Naz[22], Peyghan-Samadi-Tayedi[21] and Cagman-Karatas-Enginoglu[9] for soft topology. For studies in soft topologically algebraic structures, one can refer to Das-Majumdar-Samanta[10] for soft linear spaces and soft normed linear spaces; Das-Samanta[11] for soft metric spaces and soft inner product spaces etc..

Please notice that the above list is far from being complete and our aim with that listing is only to suggest a few papers for a beginner in each direction.

Coming to this paper, our aim here is to record some lattice theoretic properties of factorizable substructures of a product group and use them to construct crisp group for a given soft group in such a way that the complete lattice of all soft substructures of the soft group is complete epimorphic to a complete lattice of certain crisp substructures of the crisp group. Throughout the paper, proofs are left for two reasons, namely in most cases they are simple or straightforward but a little involving and secondly to minimize the size of the document. However, in order to make the document more self contained, we recall as many notions and results that are used in subsequent sections, as possible.

II. Preliminaries

In what follows we recall some basic definitions in the theory of Lattices, Groups, Factorizable subsets, Soft Sets, Soft (normal) subgroups which are used in the main results.

Definitions and Statements 2.1 (a) For any pair of posets P, Q and for any pair of order preserving maps $f: P \to Q$ and $g: Q \to P$, the pair (g, f) is a *Galois connection* between P and Q iff $gf \geq 1_P$ and $fg \leq 1_Q$. (b) For any set U and for any family $\mathcal{F}$ contained in $P(U)$, $\mathcal{F}$ is a closure system on U iff it containes U and is closed under intersections.

The following Example shows that $(X)_{\mathcal{F}}$ varies as $\mathcal{F}$ varies:

Example: Let U=$\{1,2,3,4,5,6\}$, $X = \{2,4\}$, $\mathcal{F}_1 = \{U, \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$ and $\mathcal{F}_2 = \{U, \{2,4,6,8\}, \{2,4,6\}\}$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are closure systems on U, $(X)_{\mathcal{F}_1} = \cap_{X \subseteq Y, Y \in \mathcal{F}_1} Y = U$ and $(X)_{\mathcal{F}_2} = \cap_{X \subseteq Y, Y \in \mathcal{F}_2} Y = \{2,4,6\}$.

Lattices (d) For any non-empty subset S of a meet complete poset L with the greatest element 1_L, for any subset $\phi \neq S \subseteq L$ one can define $\bar{\vee} S = \wedge\{\alpha \in L/\alpha \wedge \beta = \beta \ \ for\, all \ \ \beta \in S\}$. Then L is a complete lattice, where the join is given by $\bar{\vee}$. For any meet complete poset L with the greatest element 1_L, the join defined as above is called the *meet induced join* on L.

Groups: (e) Whenever G is a group and H is a (normal) subgroup of G we write ($H \leq_n G$) $H \leq_s G$ and ($\mathcal{S}_n(G)$) $\mathcal{S}_s(G)$ denotes the set of all (normal) subgroups of G. (f) For any index set I and for any family of groups $(G_i)_{i\in I}$, $\Pi_{i\in I} G_i = G$ is a group under pointwise multiplication called the product group.

(g) In any group G, (1) $\mathcal{S}_*(G)$ is a closure system on G as arbitrary intersection of (normal) subgroups is a (normal) subgroup and G is a (normal) subgroups of itself. (2) For any sub collection $\mathcal{C}$ of ($\mathcal{S}_n(G)$) $\mathcal{S}_s(G)$ that is a closure system on G and for any subset X of G, the intersection of all (normal) subgroups from $\mathcal{C}$ containing X, which in fact is non-empty as G is in $\mathcal{C}$ and containes X, is (i) a (normal) subgroup of G belonging to $\mathcal{C}$ (ii) the smallest (normal) subgroup from $\mathcal{C}$ containing X and (iii) it is unique in $\mathcal{C}$ with respect to (i) and (ii) (3) For any subset X of a group G and for any closure system $\mathcal{C}$ on G contained in $\mathcal{S}_*(G)$, the unique smallest (normal) subgroup in $\mathcal{C}$ containing X defined as in (g)(2) above is called the (normal) subgroup generated by X with respect to $\mathcal{C}$ and is denoted by $(X)_{\mathcal{C}}$. Observe that when $\mathcal{C}$ is all of ($\mathcal{S}_n(G)$) $\mathcal{S}_s(G)$, for any X contained G, $(X)_{\mathcal{C}}$ is nothing but the (normal) subgroups generated by X in G and is denoted by simply, ($(X)_{n,G}$) $(X)_{s,G}$.

Notice that, if A is a (normal) subgroup of G then ($(A)_{n,G}$) $(A)_{s,G}$ is itself.

(h) For any group G, the following are true:

(1) For any pair of (normal) subgroups A, B of G, A is a (normal) subgroup of B iff A is a subset of B.

(2) Whenever $* = s, n$ for any pair of subsets, A, B of G such that $A \subseteq B$, $(A)_{*,G} \leq_* (B)_{*,G}$.

(3) Whenever $* = s, n$ for any (normal) subgroup B of G and for any subset A of B, ($(A)_{n,G}$) $(A)_{s,G}$ is a (normal) subgroup of B.

(i) For any group G and for any closure system $\mathcal{C}$ contained in $\mathcal{S}_*(G)$, whenever $* = s, n$, $\mathcal{C}$ is a complete lattice with

(1) the partial order $\preceq_{\mathcal{C}}$ on $\mathcal{C}$ defined by $\preceq_{\mathcal{C}} = \{(A, B)/A, B \in \mathcal{C}, A \leq_* B\}$.

(2) the largest and the least elements in $\mathcal{C}$ are G and $\cap\, C$ respectively.

For any family $(A_i)_{i\in I}$ in $\mathcal{C}$,

(3) $\wedge_{i\in I} A_i = \cap_{i\in I} A_i$

(4) $\vee_{i\in I} A_i = \bar{\vee}_{i\in I} A_i$, where $\bar{\vee}$ is the meet induced join in $\mathcal{C}$. In fact, $\bar{\vee}_{i\in I} A_i = (\cup_{i\in I} A_i)_{\mathcal{C}} = \cap_{A_i \leq_* B, \forall i\in I, B\in\mathcal{C}} B$.

(5) $\preceq_{\mathcal{C}} = \subseteq |(\mathcal{C} \times \mathcal{C})$

Observe that from (i)(1) it follows that $A \preceq_{\mathcal{C}} B$ iff $A \leq_* B$ iff $A \subseteq B$ for all A, B in $\mathcal{C}$.

Factorizable Subsets (j) Let I be an index set, $(U_i)_{i\in I}$ be a family of sets such that $\Pi_{i\in I} U_i = U$ and $P(U)$ be the power set of U. Then a set $A \in P(U)$ is a *factorizable subset* or simply an *f-subset* of U iff $A = \Pi_{i\in I} A_i$ and $A_i \subseteq U_i$ for all $i \in I$. Notice that the empty set ϕ is trivially a factorizable subset of U and whenever $A = \Pi_{i\in I} A_i$, by the Axiom of Choice, $A \neq \phi$ iff $A_i \neq \phi$ for all $i \in I$. Thus, the set of all factorizable subsets of U, denoted by $F(U)$, is defined by $F(U) = \{A \in P(U)/\phi \neq A = \Pi_{i\in I} A_i, A_i \subseteq U_i \ \ for\, all \ \ i \in I\} \cup \{\phi\}$.

(k) For any pair of index sets I, J, for any pair of families of sets $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ and for any family of sets $(A_{j,i})_{(j,i)\in J\times I}$ the following are true: (1) $A_i \subseteq B_i$ for all $i \in I$ implies $\Pi_{i\in I} A_i \subseteq \Pi_{i\in I} B_i$. However, converse is also true whenever $\Pi_{i\in I} A_i \neq \phi$. Notice that this implies (i) $\phi \neq \Pi_{i\in I} A_i = \Pi_{i\in I} B_i$ implies $A_i = B_i$ for all $i \in I$ (ii) for all non-empty factorizable subsets A of U, the factorization of A as the Cartesian product $A = \Pi_{i\in I} A_i$ with $A_i \subseteq U_i$ for all $i \in I$, is unique. Obviously, the empty set can have different factorizations. (2) $\cap_{j\in J} (\Pi_{i\in I} A_{j,i}) = \Pi_{i\in I} (\cap_{j\in J} A_{j,i})$(3) $\cup_{j\in J} (\Pi_{i\in I} A_{j,i}) \subseteq \Pi_{i\in I} (\cup_{j\in J} A_{j,i})$.

Soft Sets In what follows we recall the following basic definitions from the Soft Set Theory which are used in the main results: (l) [17] Let U be a universal set, $P(U)$ be the power set of U and E be a set of parameters. A pair (F, E) is called a *soft set* over U iff $F: E \rightarrow P(U)$ is a mapping defined by for each $e \in E$, $F(e)$ is a subset of U. In other words, a soft set over U is a parametrized family of subsets of U.

Notice that a collective presentation of the notions, soft sets and generalized soft set or simply gs-sets raised some serious notational conflicts and to fix the same Murthy-Maheswari[19] deviated from the above notation for a soft set and adapted the following notation for convenience as follows:

Let U be a universal set. A typical *soft set* over U is an ordered pair (Bold Times New Roman) $\mathbf{S} = (\sigma_S, S)$, where S is a set of *parameters*, called the *underlying parameter set* for **S,** $P(U)$ is the power set of U and $\sigma_S : S \to P(U)$ is a map, called the *underlying set valued map* for $\mathbf{S}$. Some times σ_S is also called the *soft structure* on $\mathbf{S}$.

(m) [4] The *empty* soft set over U is a soft set with the empty parameter set, denoted by $\Phi = (\sigma_\phi, \phi)$. Clearly, it is unique. (n) [3] A soft set $\mathbf{S}$ over U is said to be a *whole* soft set, denoted by $\mathbf{U}_S$, iff $\sigma_S s = U$ for all $s \in S$. (o) [4] A soft set $\mathbf{S}$ over U is said to be a *null* soft set, denoted by Φ_S, iff $\sigma_S s = \phi$, the empty set, for all $s \in S$. Notice that $\Phi_\phi = \phi$, the empty soft (sub) set.

For any pair of soft sets **A,B** over U,

(p) [20] $\mathbf{A}$ is a *soft subset* of $\mathbf{B}$, denoted by $\mathbf{A} \subseteq \mathbf{B}$, iff (i) $A \subseteq B$ (ii) $\sigma_A a \subseteq \sigma_B a$ for all $a \in A$. The set of *all soft subsets* of $\mathbf{B}$ is denoted by $\mathcal{S}_U(\mathbf{B})$

(q) The following are easy to see:

(i) Always the empty soft set Φ is a soft subset of every soft set $\mathbf{A}$

(ii) $\mathbf{A} = \mathbf{B}$ iff $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$ iff $A = B$ and $\sigma_A a = \sigma_B a$ for all $a \in A$.

(r) For any family of soft subsets $(\mathbf{A}_i)_{i\in I}$ of $\mathbf{S}$,

(i) the *soft union* of $(\mathbf{A}_i)_{i\in I}$, denoted by $\cup_{i\in I} \mathbf{A}_i$, is defined by the soft set $\mathbf{A}$, where (i) $A = \cup_{i\in I} A_i$ (ii) $\sigma_A a = \cup_{i\in I_a} \sigma_{A_i} a$, where $I_a = \{i \in I / a \in A_i\}$, for all $a \in A$

(ii) the *soft intersection* of $(\mathbf{A}_i)_{i\in I}$, denoted by $\cap_{i\in I} \mathbf{A}_i$, is defined by the soft set $\mathbf{A}$, where (i) $A = \cap_{i\in I} A_i$ (ii) $\sigma_A a = \cap_{i\in I} \sigma_{A_i} a$ for all $a \in A$.

Notice that $\cap_{i\in I} \mathbf{A}_i$ can become empty resulting the soft intersection in the empty soft set.

Soft Groups In this section we first recall the existing notions of all soft substructures of a soft Group.

(s) [2] Let (F, A) be a soft set over G. Then (F, A) is said to be a soft group over G if, and only if $F(x) \leq G$ for all $x \in A$. (t) Let (F, A) and (H, K) be two soft groups over G. Then (H, K) is a soft subgroup of (F, A), written as $(H, K) \leq (F, A)$, if (1) $K \subseteq A$ (2) $H(x) \leq F(x)$ for all $x \in K$. (u) Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then we say that (H, B) is a normal soft subgroup of (F, A), written $(H, B) \trianglelefteq (F, A)$, if $H(x)$ is a normal subgroup of $F(x)$ i.e., $H(x) \trianglelefteq F(x)$, for all $x \in B$.

(v) Generalizing the definitions of soft substructures in [2], a soft set $\mathbf{E}$ over a group G is said to be a *soft group* over G iff $\sigma_E e$ is a subgroup of G for all $e \in E$. Consequently, for us (i) The empty soft set Φ over G is *trivially* a soft group over G because there is *no* $e \in \phi$ such that $\sigma_\phi e$ is *not* a subgroup of G and it is called the *empty* soft subgroup over G. Clearly, it is unique. (ii) A soft group $\mathbf{E}$ over G is said to be a *whole* soft group iff $\sigma_E e = G$ for all $e \in E$.

(iii) For any pair of soft subsets **A, B** of a soft group $\mathbf{E}$ over G, $\mathbf{A}$ is a soft (normal) subgroup of $\mathbf{B}$ iff $A \subseteq B$ and $\sigma_A e$ is a (normal) subgroup of $\sigma_B e$ for all $e \in A$.

The set of *all soft (normal) subgroups* of $\mathbf{E}$ is denoted by ($\mathcal{S}_n(\mathbf{E})$) $\mathcal{S}_s(\mathbf{E})$.

Theorem 2.2 For any soft group $\mathbf{E}$, whenever $* = \mathrm{s}, \mathrm{n}$, the set $\mathcal{S}_*(\mathbf{E})$ is a complete lattice with

(1) the partial ordering defined by: for any $\mathbf{A}, \mathbf{B} \in \mathcal{S}_*(\mathbf{E})$, $\mathbf{A} \leq \mathbf{B}$ iff $\mathbf{A} \leq_* \mathbf{B}$

(2) the largest and the least elements in $\mathcal{S}_*(\mathbf{E})$ are $\mathbf{E}$ and Φ respectively

For any family $(\mathbf{A}_i)_{i\in I}$ in $\mathcal{S}_*(\mathbf{E})$,

(3) $\wedge_{i\in I} \mathbf{A}_i = \cap_{i\in I} \mathbf{A}_i$

(4) $\vee_{i\in I} \mathbf{A}_i = \overline{\nabla}_{i\in I} \mathbf{A}_i$, where $\overline{\nabla}$ is the meet induced join in $\mathcal{S}_*(\mathbf{E})$. In fact, $\overline{\nabla}_{i\in I} \mathbf{A}_i = \mathbf{A}$ where $A = \cup_{i\in I} A_i$, $\sigma_A e = (\cup_{i\in I_e} \sigma_{A_i} e)_{*, \sigma_E e}$ for all $e \in E$, where $I_e = \{i \in I / e \in A_i\}$.

III. Main Theorems

In this section first we introduce and study factorizable substructures of a product group, next we introduce and study extended substructures of a soft group and finally for any soft group we construct crisp group in such a way that the complete lattice of all soft substructures of the soft group is complete epimorphic to a complete lattice of certain crisp substructures of the crisp group.

Factorizable substructures of a Product Group:
In this section we introduce the notions of factorizable (normal) subgroup of a product group and study some of their lattice theoretic properties.
Definitions and Statements 3.1 (a) Let us recall for any index set I and for any family of groups $(G_i)_{i\in I}$, $\Pi_{i\in I} G_i$ = G is a group under pointwise multiplication called the product group and the same applies throughout this paper for all products of groups.
(b) Let $F(G)$ be the set of all f-subsets of G.
An f-subset A of G is said to be an f-(normal) subgroup of G iff A_i is a (normal) subgroup of G_i for all $i \in I$ and the multiplication in A is the one induced by the multiplications in A_i.
Notice that (e) is trivially an f-(normal) subgroup of G, where $e = e_G \in G$ is such that $e_G(i) = e_{G_i}$ for all $i \in I$.
Thus, the set of all f-subgroup of G, denoted by $F_s(G)$, is defined by $F_s(G)$ = $\{A \in F(G)/A = \Pi_{i\in I} A_i,\ A_i$ *is a subgroup of* G_i *for all* $i \in I\}$ and the set of all f-normal subgroup of G, denoted by $F_n(G)$, is defined by $F_n(G) = \{A \in F(G)/A = \Pi_{i\in I} A_i,\ A_i$ *is a normal subgroup of* G_i *for all* $i \in I\}$.

Lemma 3.2 For any index set I, for any family of groups $(G_i)_{i\in I}$ such that $\Pi_{i\in I} G_i$ = G and for any family of sets $(A_i)_{i\in I}$ such that $A_i \subseteq G_i$ for all $i \in I$, the following are true:
(1) A_i is a (normal) subgroup of G_i for all $i \in I$ implies $\Pi_{i\in I} A_i$ is a (normal) subgroup of $\Pi_{i\in I} G_i$.
(2) $\Pi_{i\in I} A_i$ is a (normal) subgroup of $\Pi_{i\in I} G_i$ implies A_i is a (normal) subgroup of G_i for all $i \in I$.
Proof: It follow from the definitions of (normal) subgroup and 2.1(k).

The following example shows that (2) of the above lemma is *not* true if the multiplication in G is not pointwise.
Example: (1) Let $G_1 = \{0,1,2,3\}$, $G_2 = \{e\}$ and $G = G_1 X G_2$ be the groups with the following cayley tables:

$\cdot_{G_1\times G_2}$	(0,e)	(1,e)	(2,e)	(3,e)
(0,e)	(0,e)	(1,e)	(2,e)	(3,e)
(1,e)	(1,e)	(0,e)	(3,e)	(2,e)
(2,e)	(2,e)	(3,e)	(1,e)	(0,e)
(3,e)	(3,e)	(2,e)	(0,e)	(1,e)

$\cdot_{G_1}$	0	1	2	3
	0	1	2	3
	1	2	3	0
	2	3	0	1
	3	0	1	2

$\cdot_{G_2}$	e
e	e

Clearly, the multiplication on G is *not* pointwise.
Let $A = \{(0,e),(1,e)\} = \{0,1\} \times \{e\} = A_1 \times A_2$. Then A is a subgroup of G but A_1 is not subgroup of G_1.
(2) $G = G_1 \times G_2$ is an abelian group as the multiplication table is symmetric consequently $A = A_1 \times A_2$ is a normal subgroup of G. However A_1 is not even a subgroup of G.

Lemma 3.3 For any pair of index sets I, J, for any family of groups $(G_i)_{i\in I}$ such that $\Pi_{i\in I} G_i = G$ and for any family $(A_{j,i})_{j\in J}$ of (normal) subgroups of G_i for all $i \in I$, the following are true:
(1) $\cap_{j\in J} (\Pi_{i\in I} A_{j,i}) = \Pi_{i\in I}(\cap_{j\in J} A_{j,i})$. In other words, arbitrary intersection of f-(normal) subgroups of G is an f-(normal) subgroup of G.
(2) $(\cup_{j\in J} (\Pi_{i\in I} A_{j,i}))_{F_*(G)} = \Pi_{i\in I}(\cup_{j\in J} A_{j,i})_{S_*(G_i)}$, whenever $* = n, s$.
Proof*:* (1) follow from 3.2(1), 2.1(g)(1) and 2.1(h) (2) follow from 3.2(1) and 2.1(g).

Lemma 3.4 For any family of groups $(G_i)_{i\in I}$ such that $\Pi_{i\in I} G_i = G$, the collection $(F_n(G))$ $F_s(G)$ is a closure system and for any $X \in P(G)$, there exist unique smallest f-(normal) subgroups in $(F_n(G))$ $F_s(G)$ containing X and is denoted by $((X)_{F_n(G)})$ $(X)_{F_s(G)}$.
Proof: It follow from 2.1(b),(c) and 3.3.

Theorem 3.5 Whenever $* = s, n$ for any index set I and for any family of groups $(G_i)_{i\in I}$ such that $\Pi_{i\in I} G_i = G$, the set $F_*(G)$ is a complete lattice with
(1) the partial order $\preceq_{F_*(G)}$ on $F_*(G)$ defined by $\preceq_{F_*(G)} = \{(A,B)/A,B \in F_*(G),\ A_i \subseteq B_i$ *for all* $i \in I\}$.
(2) the largest and the least elements in $F_*(G)$ are G and (e) respectively.

For any family $(A_j)_{j\in J}$ in $F_*(G)$,
(3) $\wedge_{j\in J} A_j = \cap_{j\in J} A_j$.
(4) $\vee_{j\in J} A_j = \overline{\nabla}_{j\in J} A_j$, where $\overline{\nabla}$ is the meet induced join. In fact, $\overline{\nabla}_{j\in J} A_j = \Pi_{i\in I}(\cup_{j\in J} A_{j,i})_{\mathcal{S}_*(G_i)}$, where $A_j = \Pi_{i\in I} A_{j,i}$.
(5) $\preceq_{\mathcal{S}_*(G)} \cap (F_*(G) \times F_*(G)) = \Pi_{i\in I} \preceq_{\mathcal{S}_*(G_i)} = \preceq_{F_*(G)}$.
Proof: It is straightforward.

Corollary 3.6 For any index set I and for any group G, whenever $* = \mathrm{s}, \mathrm{n}$ the set $\mathrm{F}_*(\mathrm{G}^{\mathrm{I}})$ is a complete lattice with
(1) the partial order $\preceq_{F_*(G^I)}$ on $F_*(G^I)$ defined by $\preceq_{F_*(G^I)} = \{(A,B)/A,B \in F_*(G^I), A_i \subseteq B_i \ \ for\ all\ \ i \in I\}$.
(2) the largest and the least elements in $F_*(G^I)$ are G^I and (e) respectively.
For any family $(A_j)_{j\in J}$ in $F_*(G^I)$,
(3) $\wedge_{j\in J} A_j = \cap_{j\in J} A_j$.
(4) $\vee_{j\in J} A_j = \overline{\nabla}_{j\in J} A_j$, where $\overline{\nabla}$ is the meet induced join. In fact, $\overline{\nabla}_{j\in J} A_j = \Pi_{i\in I}(\cup_{j\in J} A_{j,i})_{\mathcal{S}_*(G)}$, where $A_j = \Pi_{i\in I} A_{j,i}$.
(5) $\preceq_{\mathcal{S}_*(G^I)} \cap (F_*(G^I) \times F_*(G^I)) = \Pi_{i\in I} \preceq_{\mathcal{S}_*(G)} = \preceq_{F_*(G^I)}$
Proof: It follows from Theorem 3.5.

Extended soft substructures of a soft group:
In this section we introduce the notions of extended soft substructures like, extended soft (normal) subgroups for the soft substructures like, soft (normal) subgroup of a soft group and study the lattice homomorphic properties of the operators induced by the same. Throughout this and the sections here after, all our soft groups are over the fixed group G and so, we drop the words, "over G" here onwards.
Definitions and Statements 3.7 (a) For any soft (normal) subgroup $\mathbf{A}$ of a soft group $\mathbf{E}$, the extended soft (normal) subgroup or simply the es-(normal) subgroup for $\mathbf{A}$ denoted by $\mathbf{A}'$, is defined by $\mathrm{A}' = \mathrm{E}$ and for all $e \in \mathrm{E}$.

$$\sigma_{A'}e = \begin{cases} \sigma_A e & \text{if } e \in A \\ (e_G) & \text{if } e \in E - A \end{cases}$$

Whenever $* = s, n$, the set of all extended soft (normal) subgroups of $\mathbf{E}$ is denoted by $\mathcal{S}_*(\mathbf{E})'$.
(b) Observe that, (i) for any soft group $\mathbf{E}$, the es-group for $\mathbf{E}$ is given by $\mathbf{E}'$, where $\mathrm{E}' = E$ and for all $e \in E$. $\sigma_{E'}e = \sigma_E e$ if $e \in E$ or $\mathbf{E}' = \mathbf{E}$. (ii) $\Phi' = \mathbf{B}$ where $B = E$ and $\sigma_B b = (e_G)$ for all $b \in B$.
(c) Observe that, for any soft subset $\mathbf{A}$ of a soft group $\mathbf{E}$, $\mathbf{A}$ is soft (normal) subgroup iff $\mathbf{A}'$ is a soft (normal) subgroup.
(d) In view of (c) above whenever $* = s, n$ the set $\mathcal{S}_*(\mathbf{E})'$ is precisely the set of all soft (normal) subgroups of $\mathbf{E}$ whose parameter set is all of E.

Theorem 3.8 For any soft group $\mathbf{E}$, whenever $* = \mathrm{s}, \mathrm{n}$ the map $\epsilon_* : \mathcal{S}_*(\mathbf{E}) \to \mathcal{S}_*(\mathbf{E})'$ defined by for any $\mathbf{A} \in \mathcal{S}_*(\mathbf{E})$, $\epsilon_*\mathbf{A} = \mathbf{A}'$ being the es-(normal) subgroup for $\mathbf{A}$, satisfies the following properties:
1) The map ϵ_* is onto.
2) For any $\mathbf{A}, \mathbf{B} \in \mathcal{S}_*(\mathbf{E})$, $\mathbf{A} \leq \mathbf{B}$ implies $\epsilon_*\mathbf{A} \leq \epsilon_*\mathbf{B}$.
For any family $(\mathbf{A}_i)_{i\in I}$ in $\mathcal{S}_*(\mathbf{E})$,
3) $\epsilon_*(\cap_{i\in I} \mathbf{A}_i) = \cap_{i\in I} \epsilon_*\mathbf{A}_i$
4) $\epsilon_*(\overline{\nabla}_{i\in I} \mathbf{A}_i) = \overline{\nabla}\, \epsilon_*\mathbf{A}_i$
5) The map ϵ_* is a complete epimorphism.
Proof: It is straightforward and follows from 2.2 and 3.7.

The following Example shows that in the above Theorem, whenever $* = s, n$, the map ϵ_* is *not* necessarily one-one.
Example: Let $G = \mathbb{Z}_3$ and $\mathbf{E} = (\{(e_1, G), (e_2, G)\}, \{e_1, e_2\})$ be the whole soft group. Then $\mathbf{E} = \mathbf{E}'$. Let $\mathbf{A_1} = (\{(e_1, (0)), \{e_1\})$ and $\mathbf{A_2} = (\{(e_2, (0)), \{e_2\})$ be the soft (normal) subgroups of $\mathbf{E}$.
Then $\mathbf{A'_1} = (\{(e_1, (0), (e_2, (0))\}, \{e_1, e_2\}) = \mathbf{A'_2}$ are soft (normal) subgroups of $\mathbf{E}$. Now $\epsilon_*\mathbf{A_1} = \mathbf{A'_1} = \mathbf{A'_2} = \epsilon_*\mathbf{A_2}$ but $\mathbf{A_1} \neq \mathbf{A_2}$.

Therefore the map ϵ_* is not a monomorphism.

Theorem 3.9 For any soft group $\mathbf{E}$, whenever $* = s, n$, the set $\mathcal{S}_*(\mathbf{E})'$ is a complete sublattice of the complete lattice $\mathcal{S}_*(\mathbf{E})$ with

(1) the induced partial ordering defined from the super poset $\mathcal{S}_*(\mathbf{E})$

(2) the largest and the least elements in $\mathcal{S}_*(\mathbf{E})'$ are $\mathbf{E}$ and Φ' respectively.

For any family $(\mathbf{A}'_1)_{i\in I}$ in $\mathcal{S}_*(\mathbf{E})'$,

(3) $\wedge_{i\in I} \mathbf{A}'_1 = \cap_{i\in I} \mathbf{A}'_1$

(4) $\vee_{i\in I} \mathbf{A}'_1 = \overline{\vee}_{i\in I} \mathbf{A}'_1$ where $\overline{\vee}$ is the meet induced join in $\mathcal{S}_*(\mathbf{E})$. In fact $\overline{\vee}_{i\in I} \mathbf{A}'_1 = \mathbf{A}'$ where $A' = E$ and $\sigma_{A'}e = (\cup_{i\in I} \sigma_{A'} e)_{*,\sigma_E e}$.

(5) In fact, $\mathcal{S}_*(\mathbf{E})'$ is a complete filter (generated by Φ' in $\mathcal{S}_*(\mathbf{E})$) of $\mathcal{S}_*(\mathbf{E})$

(6) Thus the inclusion map $i_*: \mathcal{S}_*(\mathbf{E})' \to \mathcal{S}_*(\mathbf{E})$ is a complete monomorphism.

Proof: It is straightforward.

Representation of soft substructures:

In this section we introduce the notions of associated product group for a soft (normal) subgroup, f-(normal) subgroups of the associated product group and study some of their lattice theoretic properties. Further, we construct a Galois connection between the complete lattice of all es-substructures of a soft group $\mathbf{E}$ and the complete lattice of all f-substructures of the associated product group $\Pi\mathbf{E}$.

Definitions and Statements 3.10 (a) Let us recall that for any index set I and for any family of groups $(\mathrm{G_i})_{\mathrm{i}\in\mathrm{I}}$, $\Pi_{\mathrm{i}\in\mathrm{I}}\mathrm{G_i}$ is a group under pointwise multiplication called the product group and the same applies throughout this section for all products of groups.

(b) For any soft group $\mathbf{E}$, the associated product group for $\mathbf{E}$ under pointwise multiplication, denoted by $\Pi\mathbf{E}$, is defined by $\Pi\mathbf{E} = \Pi_{e\in E}\sigma_E e$.

(c) Let $\mathbf{E}$ be a soft group, ΠE be the associated product group for $\mathbf{E}$ and $F(\Pi\mathbf{E})$ be the set of all f-subsets of $\Pi\mathbf{E}$.

An f-subset X of $\Pi\mathbf{E}$ is said to be an f-(normal) subgroup of $\Pi\mathbf{E}$ iff X_e is a (normal) subgroup of $\sigma_E e$ for all $e \in E$ and the multiplication on X is the one induced by the multiplications on X_e.

whenever $X = \Pi_{e\in E}X_e$, by the axiom of choice, $X \neq \phi$ if and only if $X_e \neq \phi$ for all $e \in E$.

The set of all f-(normal) subgroups of $\Pi\mathbf{E}$ denoted by ($F_n(\Pi\mathbf{E})$) $F_s(\Pi\mathbf{E})$ and is given by ($F_n(\Pi\mathbf{E})$ = $\{X \in F(\Pi\mathbf{E})/X = \Pi_{e\in E}X_e$ *and* X_e *is a normal subgroup of* $\sigma_E e$ *for all* $e \in E\}$) $F_s(\Pi\mathbf{E}) = \{X \in F(\Pi\mathbf{E})/X = \Pi_{e\in E}X_e$ *and* X_e *is a subgroup of* $\sigma_E e$ *for all* $e \in E\}$.

Corollary 3.11 For any soft group $\mathbf{E}$, whenever $* = s, n$ the set $F_*(\Pi\mathbf{E})$ is a complete lattice with

(1) the partial order $\leq_*$ on $F_*(\Pi\mathbf{E})$ defined by $\leq_* = \{(X, Y)/X, Y \in F_*(\Pi\mathbf{E}), X_e \leq_* Y_e$ for all $e \in E\}$.

(2) the largest and least element in $F_*(\Pi\mathbf{E})$ are $\Pi\mathbf{E}$ and (e) respectively

For any family $(X_j)_{j\in J}$ in $F_*(\Pi\mathbf{E})$,

(3) $\wedge_{j\in J} X_j = \cap_{j\in J} X_j$

(4) $\overline{\vee}_{j\in J} X_j = \Pi_{e\in E}(\cup_{j\in J} X_{j,e})_{*,\sigma_E e}$ Notice that, by 3.3(2) the RHS equals $(\cup_{j\in J} (\Pi_{e\in E}X_{j,e}))_{F_*(\Pi\mathbf{E})}$.

Proof: It follows from the Theorem 3.5.

Galois Connections Between all es-substructures and all f-substructures:

In this section we construct a Galois connection between the complete lattice of all es-substructures for all soft substructures of a soft group $\mathbf{E}$ and the complete filter of all f-substructures of $\Pi\mathbf{E}$.

Theorem 3.12 For any soft group $\mathbf{E}$, whenever $* = s, n$, the maps $\lambda_* : \mathcal{S}_*(\mathbf{E})' \to F_*(\Pi\mathbf{E})$ defined by for any $\mathbf{A}' \in \mathcal{S}_*(\mathbf{E})'$, $\lambda_*\mathbf{A}' = \Pi_{e\in E}\sigma_{A'}e$ and $\mu_* : F_*(\Pi\mathbf{E}) \to \mathcal{S}_*(\mathbf{E})'$ defined by for any $X \in F_*(\Pi\mathbf{E})$, $\mu_*X = \mathbf{C}$, where $C = E$ and $\sigma_C e = X_e$ for all $e \in E$, satisify the following properties:

(1) $\lambda_* \circ \mu_* = 1_{F_*(\Pi\mathbf{E})}$, where $1_{F_*(\Pi\mathbf{E})}$ is the identity map on $F_*(\Pi\mathbf{E})$.

(2) $\mu_* \circ \lambda_* = 1_{\mathcal{S}_*(\mathbf{E})'}$, where $1_{\mathcal{S}_*(\mathbf{E})'}$ is the identity map on $\mathcal{S}_*(\mathbf{E})'$

(3) Both λ_* and μ_* are one-one and onto.

(4) For any $\mathbf{A}', \mathbf{B}' \in \mathcal{S}_*(\mathbf{E})'$, $\mathbf{A}' \leq \mathbf{B}'$ implies $\lambda_*\mathbf{A}' \leq \lambda_*\mathbf{B}'$.

(5) For any $X, Y \in F_*(\Pi\mathbf{E})$, $X \leq Y$ implies $\mu_*X \leq \mu_*Y$

For any family $(\mathbf{A}'_1)_{i\in I}$ in $\mathcal{S}_*(\mathbf{E})'$,

(6) $\lambda_*(\cap_{i\in I} \mathbf{A}'_1) = \cap_{i\in I} \lambda_*\mathbf{A}'_1$
(7) $\lambda_*(\overline{\vee}_{i\in I} \mathbf{A}'_1) = \overline{\vee}_{i\in I} \lambda_*\mathbf{A}'_1$
For any family $(X_i)_{i\in I}$ in $F_*(\Pi\mathbf{E})$
(8) $\mu_*(\cap_{i\in I} X_i) = \cap_{i\in I} \mu_* X_i$
(9) $\mu_*(\overline{\vee}_{i\in I} X_i) = \overline{\vee}_{i\in I} \mu_* X_i$
(10) The map λ_* is a complete isomorphism.
(11) The map μ_* is a complete isomorphism.
Proof*:* It is straightforward.

Theorem 3.13 For any soft group $\mathbf{E}$, whenever $* = s, n$ the map $\lambda_* \circ \epsilon_*: \mathcal{S}_*(\mathbf{E}) \to F_*(\Pi\mathbf{E})$ defined by for any $\mathbf{A} \in \mathcal{S}_*(\mathbf{E})$, $(\lambda_* \circ \epsilon_*)(\mathbf{A}) = \lambda_*(\epsilon_*\mathbf{A}) = \lambda_*(\mathbf{A}') = \Pi_{e\in E}\sigma_{A'}e$, satisfies the following properties
(1) The map $\lambda_* \circ \epsilon_*$ is onto.
For any family $(\mathbf{A}_i)_{i\in I}$ in $\mathcal{S}_*(\mathbf{E})$,
(2) $(\lambda_* \circ \epsilon_*)(\cap_{i\in I} \mathbf{A}_i) = \cap_{i\in I} (\lambda_* \circ \epsilon_*)(\mathbf{A}_i)$
(3) $(\lambda_* \circ \epsilon_*)(\overline{\vee}_{i\in I} \mathbf{A}_i) = \overline{\vee}_{i\in I} (\lambda_* \circ \epsilon_*)(\mathbf{A}_i)$
(4) The map $\lambda_* \circ \epsilon_*$ is a epimorphism.
Proof: (1) It follows from 3.8(1) and 3.12(3)
(2) It follows from 3.8(3) and 3.12(6)
(3) It follows from 3.8(4) and 3.12 (7)
(4) It follows from (1),(2) and (3).

Corollary 3.14 For any soft group $\mathbf{E}$, there is a crisp group G such that the complete lattice of all soft (normal) subgroups of $\mathbf{E}$ is complete epimorphic to a complete lattice of certain (normal) subgroups of G, where the joins in the former and the later complete lattices are the meet induced joins.
Proof: It follow from 3.13.

Corollary 3.15 For any soft group $\mathbf{E}$, whenever $* = s, n$, the maps $f_* = \lambda_* \circ \epsilon_*: \mathcal{S}_*(\mathbf{E}) \to F_*(\Pi\mathbf{E})$ and $g_* = i_* \circ \mu_*$: $F_*(\Pi\mathbf{E}) \to \mathcal{S}_*(\mathbf{E})$ define a Galois connection between $\mathcal{S}_*(\mathbf{E})$ and $F_*(\Pi\mathbf{E})$.
Proof: It follows from 3.13, 3.12 and 3.9.

IV. Conclusion

In this paper, for any soft group $\mathbf{E}$ over a group, we constructed a crisp group G such that the complete lattice of all soft (normal) subgroups of $\mathbf{E}$ is complete epimorphic to a complete lattice of certain (normal) subgroups of G, where the joins in the former and the later complete lattices are the meet induced joins.

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