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Cauchy-Schwartz Inequality of Euclidean and Probability Space

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Abstract: *Cauchy-Schwartz's inequality is a kind of important inequality widely used in mathematics, and often serves as an important basis to bridge the assumption between conditions and conclusions.The n-dimensional Euclidean space provides a strong theoretical basis for the establishment of Cauchy-Schwartz's inequality, that is, the Cauchy-Schwartz inequality can be establishedin the n-dimensional Euclidean space to establish the n-dimensional Euclidean space as the Cauchy-Schwartz inequality. The establishment provides a strong theoretical basis and can provide rigorous proof of Schwartz's inequality in probability space.*
Keywords: *Cauchy-Schwartz inequality, Euclidean space, Probability space*

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I. Introduction

Cauchy inequality was obtained by the great mathematician Cauchy when studying the problem of "flow number" in mathematical analysis. However, from a historical point of view, this inequality should be called Cauchy-Schwartz inequality, because it is the last two mathematicians who generalize each other independently in integral science to apply this inequality to the point of near perfection. Cauchy inequality is very important, flexible and ingenious application of it, can solve some of the more difficult problems. Cauchy inequality is obtained by proving inequality, Euclidean space and even regression equation in statistics to apply.

II. Cauchy-Schwartz Inequality and its Extension in Euclidean Space

Definition 1.1 (Euclidean space): Set to be the vector space on the real number field. If there is a mapping $f:V\times V\to R$, it is recorded as $(\alpha,\beta)\to f(\alpha,\beta)$. It will be recorded as having three properties:(1) symmetry: $(\alpha,\beta)=(\beta,\alpha),\forall\alpha,\beta\in V;$

(2) Linearity: $(k_1\alpha_1+k_2\alpha_2,\beta)=k_1(\alpha_1,\beta)+k_2(\alpha_2,\beta),\forall k_1,k_2\in R,\forall\alpha_1,\alpha_2,\beta\in V;$

(3) Non-negative: there is $(\alpha,\alpha)\geq 0$ if $\forall\alpha\in V$ and $\alpha=0$,$(\alpha,\alpha)=0$ only then.

Theorem 1.2: In the n-dimensional Euclidean-Schwartz inequality space, assume $\alpha=(a_1,a_2,...,a_n)$,$\beta=(b_1,b_2,...,b_n)$ are N-dimensional vectors, then there are

$$|(\alpha,\beta)|\leq\|\alpha\|\cdot\|\beta\| \qquad (1)$$

The equal sign holds if and only if it is related to linearity.

Proof: If $\alpha=0$, then $|(0,\beta)|=0$, the equal sign is established; If α ,β are linear correlation , let's set $\alpha=k\beta$ up, then $|(\alpha,\beta)|=|(k\beta,\beta)|=|k(\beta,\beta)|=|k|\|\beta\|^2$,on the other hand, because $\|\alpha\|\cdot\|\beta\|=\|k\beta\|\cdot\|\beta\|=|k|\|\beta\|\cdot\|\beta\|=|k|\|\beta\|^2$, So there is always $|(\alpha,\beta)|\leq\|\alpha\|\cdot\|\beta\|$; If α ,β linearity is irrelevant, set $\gamma=\alpha+k\beta$, for $\forall k\in F,\gamma\neq 0$, Then $0<(\gamma,\gamma)=(\alpha,\alpha)+2k(\alpha,\beta)+k^2(\beta,\beta).$ Especially order $k=-\dfrac{(\alpha,\beta)}{(\beta,\beta)}$ because it has nothing to do with linearity, so $\beta\neq 0,\ (\beta,\beta)>0,$ then there is $(\alpha,\alpha)-2\dfrac{(\alpha,\beta)}{(\beta,\beta)}(\alpha,\beta)+\dfrac{(\alpha,\beta)^2}{(\beta,\beta)}>0,$ equivalent to

$$(\alpha,\beta)^2<(\alpha,\alpha)(\beta,\beta) \qquad (2)$$

Equation (2), the Cauchy-Schwartz inequality in the famous Euclidean space is proved.

$(\alpha,\beta)^2 < (\alpha,\alpha)(\beta,\beta)$ can be rewritten into a determinant form, so

$$\begin{vmatrix} (\alpha,\alpha) & (\alpha,\beta) \\ (\beta,\alpha) & (\beta,\beta) \end{vmatrix} \geq 0$$

So we can generalize the Cauchy Schwartz inequality to n vector groups in[1] n-dimensional Euclidean space. Let any n vector groups $\alpha_1, \alpha_2, ..., \alpha_n$ in n-dimensional Euclidean space, so determinant is

$$M(\alpha_1, \alpha_2, \dots, \alpha_n) = \begin{vmatrix} (\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \dots & (\alpha_1,\alpha_n) \\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2) & \dots & (\alpha_2,\alpha_n) \\ \dots & \dots & \dots & \dots \\ (\alpha_n,\alpha_1) & (\alpha_n,\alpha_2) & \dots & (\alpha_n,\alpha_n) \end{vmatrix} \geq 0$$

The equal sign holds if and only if the vector groups are linearly related.

Proof: If linearly related, there is a set of constants $k_1, k_2, ..., k_n$ that are not all zero, making $k_1\alpha_1 + k_2\alpha_2 + ... + k_n\alpha_n = 0$. Because $(\alpha_i, 0) = 0$, $i = 1,2,\dots,n,$ therefore

$$\begin{cases} k_1(\alpha_1,\alpha_1) + k_2(\alpha_1,\alpha_2) + \dots + k_n(\alpha_1,\alpha_n) = 0 \\ k_1(\alpha_2,\alpha_1) + k_2(\alpha_2,\alpha_2) + \dots + k_n(\alpha_2,\alpha_n) = 0 \\ \qquad \dots \\ k_1(\alpha_n,\alpha_1) + k_2(\alpha_n,\alpha_2) + \dots + k_n(\alpha_n,\alpha_n) = 0 \end{cases}$$

to $k_1, k_2, ..., k_n$ for a homogeneous linear equation system of unknown quantity, the system of equations has a non-zero solution, so the determinant $M(\alpha_1, \alpha_2, ..., \alpha_n) = 0$.

If the linearity $\alpha_1, \alpha_2, ..., \alpha_n$ is irrelevant, then the orthogonal vector group $\beta_1, \beta_2, ..., \beta_n$ is available, and $(\alpha_1, \alpha_2, ..., \alpha_n) = (\beta_1, \beta_2, ..., \beta_n)T$. Particularly

$$T = \begin{pmatrix} 1 & \frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)} & \frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)} & \dots & \frac{(\alpha_n,\beta_1)}{(\beta_1,\beta_1)} \\ 0 & 1 & \frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)} & \dots & \frac{(\alpha_n,\beta_2)}{(\beta_2,\beta_2)} \\ 0 & 0 & 1 & \dots & \frac{(\alpha_n,\beta_3)}{(\beta_3,\beta_3)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Obviously $|T| = 1 \neq 0$, it is reversible, so the vector group $\alpha_1, \alpha_2, ..., \alpha_n$ is equivalent to the vector group $\beta_1, \beta_2, ..., \beta_n$, so they generate the same subspace.And because $\beta_1, \beta_2, ..., \beta_n$ is a set of orthogonal bases, so

$$M(\beta_1, \beta_2, \dots, \beta_n) = \begin{vmatrix} (\beta_1,\beta_1) & 0 & \dots & 0 \\ 0 & (\beta_2,\beta_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (\beta_n,\beta_n) \end{vmatrix} = \prod_{i=1}^{n} (\beta_i, \beta_i) > 0$$

For $\forall \alpha, \beta \in W$, let $\alpha = x_1\alpha_1 + x_2\alpha_2 + ... + x_n\alpha_n$, $\alpha = x_1'\beta_1 + x_2'\beta_2 + ... + x_n'\beta_n$

$$\beta = y_1\alpha_1 + y_2\alpha_2 + ... + y_n\alpha_n, \ \beta = y_1'\beta_1 + y_2'\beta_2 + ... + y_n'\beta_n$$

By coordinate transformation formula $\begin{pmatrix} y_1' \\ y_2' \\ \cdots \\ y_n' \end{pmatrix} = T\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix}$, $\begin{pmatrix} x_1' \\ x_2' \\ \cdots \\ x_n' \end{pmatrix} = T\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$, then

$$(\alpha,\beta) = (x_1, x_2, \ldots, x_n)A\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, (\alpha,\beta) = (x_1', x_2', \ldots, x_n')B\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix}$$

Especially

$$A = \begin{pmatrix} (\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \ldots & (\alpha_1,\alpha_n) \\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2) & \ldots & (\alpha_2,\alpha_n) \\ \ldots & \ldots & \ldots & \ldots \\ (\alpha_n,\alpha_1) & (\alpha_n,\alpha_2) & \ldots & (\alpha_n,\alpha_n) \end{pmatrix}, B = \begin{pmatrix} (\beta_1,\beta_1) & 0 & \ldots & 0 \\ 0 & (\beta_2,\beta_2) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (\beta_n,\beta_n) \end{pmatrix},$$

$$(x_1', x_2', \ldots, x_n')B\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = (x_1, x_2, \ldots, x_n)T'BT\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix} = (x_1, x_2, \ldots, x_n)A\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix}$$

Because of the arbitrariness for $\alpha, \beta \in W$, we know that $A = T'BT$ is founded , so $M(\alpha_1, \alpha_2, \ldots, \alpha_n) = |A| = |T'BT| = |B| > 0$.

III. Special Euclidean Space-Probability Space

It has been proved above that there is a Cauchy-Schwartz inequality in Euclidean space, so if the probability space[2] is Euclidean space, then the probability space has the form of Cauchy-Schwartz inequality.

Definition 2.1: When the random variables are all discrete random variables , define $(X,Y) = E(XY), \forall X, Y \in V$.

It satisfies the nature of Euclidean space(Symmetry, Linearity, Non-negative):

$\forall X, Y \in V, (X,Y) = E(XY), (Y,X) = E(YX)$ deduced $(X,Y) = (Y,X)$;

$(k_1X_1 + k_2X_2, Z) = E[(k_1X_1 + k_2X_2)Z] = k_1E(X_1Z) + k_2E(X_2Z) = k_1(X_1,Z) + k_2(X_2,Z)$;

$\forall X \in V, (X,X) = E(X^2) \geq 0$

Thus, a linear space on the number domain composed of all random variables is made into an Euclidean space with respect to the defined inner product.The Cauchy-Schwartz inequality in the Euclidean space: $(\alpha,\beta)^2 < (\alpha,\alpha)(\beta,\beta)$ know that in the probability space, for $\forall X, Y \in V$, there are $(X,Y)^2 \leq (X,X)(Y,Y)$, let $X - E(x) \in V$ and $Y - E(Y) \in V$. Assuming that its mathematical expectations exist, there are:

$(X - E(X), Y - E(Y))^2 \leq (X - E(X), X - E(X))(Y - E(Y), Y - E(Y))$, That is $[E(X - E(X))(Y - E(Y))]^2 \leq E(X - E(X))^2 E(Y - E(Y))^2$.

Thus the Cauchy-Schwartz inequality in the probability space is obtained:

$$[Cov(X,Y)]^2 \leq \sigma_X^2 \sigma_Y^2 \quad (3)$$

IV. Conclusion

In statistics, the least squares method is used when finding two undetermined coefficients of the linear trend equation[3]. The Cauchy Schwartz inequality plays a complementary role in the equation coefficient, which enhances the accuracy, scientific and rigor of the prediction model. It can be found from the above that the

Cauchy-Schwartz inequality has a wide range of practical applications in probability theory and mathematical statistics, and the n-dimensional Euclidean space provides a powerful theoretical basis for the establishment of Cauchy-Schwartz inequality.

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