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Hankel Determinant For A Subclass Of Generalized Distribution Function Involving Jackson's Derivative Operator Through Chebyshev Polynomials

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Abstract: *In this paper, the author finds the coefficient bounds and the Hankel determinant of Generalized distribution function involving Jackson' q-derivative operator using the subordination principle.*

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I. Introduction, Preliminaries And Lemma

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C\, and\ |z| < 1\}$

and normalized by the condition $f(0) = 0 = f'(0) - 1$. Also let $H \in A$ be the class of analytic univalent function in U. A function $f \in A$ is called a starlike function denoted with S^* if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{2}$$

A function $f \in A$ which maps U onto a convex domain is called convex function denoted by K if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in u \tag{3}$$

Generally, $S^*(0) = S^*$ and $k(0) = k$

A function $f \in H$ is called a close-to-convex in *U* if the range $f(U)$ is close to convex and this is the compliment of $f(U)$ which is written as the union of non-intersecting half lines.

Moreover, a function $f \in H$ is said to be close-to-convex with respect to a fixed Starlike functions *g* (*g* not necessarily normalized) denoted by C_g, if and only if

$$\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \ z \in U.$$

It is known that very close-to-convex function in *U* is also univalent in *U* (Prajapat *et al* 2015). The decompositional approach to matrix computation, one of the top ten algorithms of the 20th century had been employed by many researchers severally. These include the Hankel and Teoplitz determinants ant their applications know no bound. For example, they provide a platform on which a variety of scientific and engineering problems can be solved and furthermore, they permit reasonably simple rounding-error analysis and afford high quality software implementations. Hankel determinants play a vital role in different branches and have many applications (Pommereke, 1975).

The study of Hankel determinant began with Noonan and Thomas in 1976, when they defined the *qth* Hankel determinant of the function in (1) by:

$$H_{qn}(f)=H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \ (n,q \in N;\ a_1 = 1) \tag{4}$$

Since then, various researchers had investigated the Hankel determinant for different univalent and bi-univalent functions, excluding generalized distribution function which is a new research area in the field of geometric function theory. Beginning with Porwal (2018) who investigated the geometric properties of generalized distribution associated with univalent functions, Oladipo (2019a) and (2019b) who investigated bonds for probabilities of the generalized distribution polylogarithm and generalized distribution associated with univalent functions in conical domain respectively. In this work, we aim at filling the vacuum created by Porwal and Oladipo thus finding the Hankel determinant for the generalized distribution involving Jackxon's q-derivative operator.

Let the series $\sum_{k=0}^{\infty} a_k, a_k \geq 0, n \in N$ be convergent and its sum is denoted by *S* such that $S = \sum_{k=0}^{\infty} a_k$

(5)

We now introduce the generalized discrete probability distribution whose probability mass function is

$$p(k) = \frac{a_k}{s}, k = 0,1,2 \tag{6}$$

Obviously, $p(k)$ is a probability mass function because $p(k) \geq 0$ and $\sum_k p_k \geq 1$

We then introduce a power series whose coefficients are probabilities of the generalized distribution, that is

$$G_d(z) = z + \sum_{k=2}^{\infty} \frac{a_k - 1}{s} z^k \tag{7}$$

Applying the *q*-derivative $[k]_q$ operator as defined in (14) on (7) we

$$D_q G_d(z) = z + \sum_{k=2}^{\infty} [k]_q \frac{a_k - 1}{s} z^k \tag{8}$$

Noor (1983) determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions *f* given by (1) with bounded boundary and also studied the Hankel determinant for Bazilevic functions in Noor (1983) and sharp upper bond for $H_2(2)$ were obtained in the recent works Noor and Banny(1987) and Hayani and Owa (2010) for different classes of functions. We note in particular that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \tag{9}$$

Moreover, the Hankel determinant $H_2(1) = a_3 - a_2^2$ is the well-known Fekete-Szego functional (see details in Fekete & Szego; 1933) . Very recently, the upper bounds of $H_2(2)$ for some analytic function classes were discussed by Deniz et al (2015).

The applications of Hankel determinant have been investigated by various researchers. Examples include Wilson (1954) who studied its application in meromorphic functions and Cantor (1963) who showed its application in function of bounded characteristic in *U*, that is, a function in which a ratio of two bounded analytic functions with its Laurent series around the region having integral coefficients as rational. Pommereke (1966) investigated the Hankel determinant of a really mean p-valent functions as well as of Starlike functions and proved that the determinants of univalent functions satisfy.

$$\left|H_{q,n}(f)\right| < k\, n^{-\left(\frac{1}{2}+\beta\right)q+\frac{3}{2}} \text{ for } n = 1,2,\ldots) \text{ and } (q = 2,3,\ldots) \tag{10}$$

where $\beta > \frac{1}{4000}$ and *k* depends only on *q*.

Hankel determinant of $f \in A$ for $q = 2$ and $n = 2$ $(H_{2,2}(f))$ is known as the second Hankel determinant and is given by

$$H_{2,2}(f) = H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2\, a_4 - a_3{}^2 \qquad (11)$$

Janteng et al (2006) have found out that $a_2\, a_4 - a_3{}^2 \le 1$ and $a_2\, a_4 - a_3{}^2 \le \frac{1}{8}$ for starlike and convex functions respectively. See also Al-Refai and Darus (2009), Al-Abbadi and Darus (1933) for details..

The third Hankel determinant $H_{3,1}(f)\, or\, H_3(1)$ is defined by

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3\left(a_2\, a_4 - a_3{}^2\right) - a_4\left(a_4 - a_2 a_3\right) + a_5\left(a_3 - a_2{}^2\right) \qquad (12)$$

for $f \in A$ and $a_1 = 1$

Applying triangle inequality, we obtain

$$\left|H_{3,1}(f)\right| \le |a_3|\left|\left(a_2\, a_4 - a_3{}^2\right)\right| - |a_4|\left|\left(a_4 - a_2 a_3\right)\right| + |a_5|\left|\left(a_3 - a_2{}^2\right)\right| \qquad (13)$$

Recently, $H_3(1)$ have been investigated by Babalola (2010) and Prajapat *et al* (2015).

In the field of Geometric function theory, various subclasses of the normalized analytic function class *A* have been studied from different points of view. The *q*-calculus as well as the fractional *q*-calculus such as fractional *q*-integral and fractional *q*-derivative operators are used to investigate several subclasses of analytic functions (Details are found in Aydogan *et al;2013,* Aldweby & Darus; 2013, Ozkan; 2016 and Polatoglu; 2016). The application of *q*-calculus which plays vital role in the theory of hyper-geometric series, quantum physics and operator theory was initiated by Jackson (1908). He was the first Mathematician who developed *q*-derivative and *q*-integral in a systematic way. Both operators play crucial role in the theory of relativity, usually encompasses two theories by Einstein, one in special relativity and the other in general relativity.

While special relativity applies to the elementary particles and their interactions, general relativity on the other hand applies to the cosmological and astrophysical realm including astronomy. Of interest is the fact that special relativity theory had rapidly become a significant and necessary tool for theorists and experimentalists in the new fields of atomic physics, nuclear physics and quantum mechanics.

Later the Hankel determinant of exponential polynomials were studied by Ehrenbog (2000) and Layman (2001) discussed one of its properties. Fekete-Szego (1933) made an early study for the estimates of $\left|a_3 - \mu\, a_2^2\right|$ when $a_1 = 1$ and μ real and obtained that if $f \in S$, then

$$\left|a_3 - \mu\, a_2{}^2\right| \le \begin{cases} 4\mu - 3, & if\ \ \mu \ge 1 \\ 1 + 2e^{\left(\frac{-2\mu}{1-\mu}\right)}, & if\ 0 \le \mu \le 1 \\ 3 - 4\mu & if\ \ \mu \le 0 \end{cases} \qquad (14)$$

In this present work, we studied the Hankel determinant of generalized distribution function involving the q-derivative (Jackson) operator via Chebyshev polynomials.

Definition 1.1 Let $q \in (0,1)$ and define

$$[k]_q = \frac{1 - q^k}{1 - q}, \text{ for } k \in N \qquad (15)$$

Definition 1.2 The Jackson's *q*-derivative of a function $f \in A,\ 0 < q < 1$ is defined as follows:

$$D_q\, f(z) = \begin{cases} f(z) - f(qz), & for\ z \ne 0 \\ f'(0), & for\ \ z = 0 \end{cases}$$

and

$$D_q{}^2\, f(z) = \Delta_q\left(\Delta_q\, f(z)\right) \qquad (16)$$

We note that $Lim_{q \to 1^-}\left(D_q\, f(z)\right) = f'(z)$ if *f* is differentiable at *z*.

From (15) and (1) we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \tag{17}$$

Definition 1.3 The symmetric q-derivative $\overline{\Delta}_q f$ of a function f given by (1) is defined as follows:

$$\overline{D}_q f(z) = \begin{cases} \dfrac{f(qz) - f\left(q^{-1}(z)\right)}{\left(q - q^{-1}\right)z} & for\ z \neq 0 \\ f'(0) & for\ z = 0 \end{cases} \tag{18}$$

From (17), we deduce that $\widetilde{D}_q z^k = [\widetilde{k}]_q\, z^{k-1}$ and a power series of

$\widetilde{D}_q f$ is $\widetilde{D}_q f(z) = 1 + \sum_{k=2}^{\infty} [\widetilde{k}]_q a_k z^{k-1}$

when f has the form (1) and the symbol $(k)_q$ denotes the number $[\widetilde{k}]_q = \dfrac{q^k - q^{-k}}{q - q^{-1}}$

Chebyshev polynomials have become increasingly important in numerical analysis from both the theoretical and practical points of view. They are sequences of orthogonal polynomials which are practically related to De-Moivres formular and which are defined recursively. There exists four kinds though, the first and second kinds $T_n(x)$ and $U_n(x)$ are well known having more results, uses and applications (Doha;1994 and Mason; 1967) In this work we shall limit ourselves to the second kind given as

$$U_k(t) = \frac{\sin(k+1)\alpha}{\sin\alpha}, t \in (-1, 1) \tag{19}$$

where k denotes the degree of the polynomial and $t = \cos\alpha$

The Chebyshev polynomials of the second kind $U_k(t); t \in (-1, 1)$ have the generating function of the form

$$H(z,t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha} z^k \left(z \in D, |t| < 1\right)$$

Note that $t = \cos\alpha, \alpha \in \left(\dfrac{-\pi}{3}, \dfrac{\pi}{3}\right)$, then

$$H(z,t) = \frac{1}{1 - 2\cos\alpha z + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha} z^k \tag{20}$$

Thus,

$$H(z,t) = 1 + 2\cos\alpha z + \left(3\cos^2\alpha - \sin^2\alpha\right)z^2 + ... \tag{21}$$

Following the relationship in Fadipe *et al* (2003), we have

$$H(z,t) = 1 + u_1(t)z + u_2(t)z^2 + ... \tag{22}$$

where $U_{k-1} = \dfrac{\sin\left(k\cos^{-1} t\right)}{\sqrt{1-t^2}}, k \in N$

are the Chebyshev polynomials of the second kind. It is known that

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t)$$

So that

$$U_1(t) = 2t,,\ U_2(t) = 4t^2 - 1 \text{ and } U_3(t) = 8t^3 - 4t \tag{23}$$

Definition 1.4 If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as $f \prec g$ or $f(z) \prec g(z)$, $z \in U$ if there exists a Schwarz function $w(z)$ which by definition is analytic in U such that

$$f(z) = g(w(z))$$

Definition 1.5 The Convolution (or Hadamard product) of two series

$f(z)=\sum_{k=0}^{\infty}a_k z^k$ and $g(z)=\sum_{k=0}^{\infty}b_k z^k$ is defined by

$$f*g(z)=\sum_{k=0}^{\infty}a_k b_k z^k$$

A Set of Lemmas:
For this study, the following existing lemmas are established:
Let P be the class of functions $p(z)$ with positive real part consisting of all analytic function $p:U\to C$ satisfying the following conditions

$p(0)=1$ and $R\left(p(z)\right)>0$

Lemma 1.1 (Pauzi, Darus and Siregar): If the function $p\in P$ is defined by

$$p(z)=1+c_1 z+c_2 z^2+c_3 z^3+...$$

then $\left|c_n\right|\le 2,\ n\in\mathbb{N}=\{1,2,3\ ...\}$

Lemma 1.2 (Pauzi, Darus and Siregar): If the function $p\in P$ is defined by

$$c(z)=1+c_1,z+c_2,z^2+c_3,z^3+...$$

then

$$2c_2=c_1{}^2+x\left(4-c_1{}^2\right)$$

and

$$4c_3=c_1{}^3+2c_1\left(4-c_1{}^2\right)x-c_1\left(4-c_1{}^2\right)x^2+2\left(4-c_1{}^2\right)\left(1-|x|^2\right)z$$

for some values of $z,\ |z|\le 1$

II. Main Results

Definition 2.1 A function $f\in A$ is said to be in class $G_d(\theta,\mu,H(z,t))$ if

$$\mathrm{Re}\left\{(1-\theta)\frac{G_d(z)}{z}+\theta G_d{}'(z)+\mu z G_d{}''(z)\right\}\prec H(z,t) \tag{24}$$

where $0\le\theta\le 1,\ \mu\ge 1$ and $H(z,t)$ is the Chebyshev polynomial

Remark 2.2. if $\theta=0$, we have

$$\mathrm{Re}\left\{\frac{G_d(z)}{z}+\mu z G_d{}''(z)\right\}\prec H(z,t) \tag{25}$$

Remark 2.3. if $\theta=1$, we have

$$\mathrm{Re}\left\{G_d{}'(z)+\mu z G_d{}''(z)\right\}\prec H(z,t) \tag{26}$$

Theorem 1: Let $f\in G_d(\theta,\mu,H(z,t))$, the generalized distribution which satisfies the subordination principle. Then

$$\left|\frac{a_1}{S}\right|\le\frac{2t}{(1+\theta+2\mu)[2]_q}, \tag{27}$$

$$\left|\frac{a_2}{S}\right|\le\frac{2t^2-1}{(1+2\theta+6\mu)[3]_q} \tag{28}$$

$$\left|\frac{a_3}{S}\right|\le\frac{4t\left(2t^2-1\right)}{(1+3\theta+12\mu)[4]_q} \tag{29}$$

Proof:

Let $f \in G_d(\theta, \mu, H(z,t))$. Then there exists a Chebyshev polynomial $H(u(z),t)$ such that

$$\mathrm{Re}\left\{(1-\theta)\frac{G_d(z)}{z} + \theta G_d{}'(z) + \mu z G_d{}''(z)\right\} \prec H(u(z),t) \tag{30}$$

where $H(z,t) = 1 + \frac{1}{2}U_1(t)c_1 z + \left(\frac{U_1(t)c_2}{2} - \frac{U_1(t)c_1{}^2}{4} + \frac{U_2(t)c_1{}^2}{4}\right)z^2$

$$+\left(\frac{U_1(t)c_3}{2} - \frac{U_1(t)c_1c_2}{2} + \frac{U_1(t)c_1{}^3}{8} + \frac{U_2(t)c_1c_2}{2} - \frac{U_2(t)c_1{}^3}{4} + \frac{U_3(t)c_1{}^3}{8}\right)z^3 + \dots \tag{31}$$

Next, define the function $p \in P$ by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + ... \tag{32}$$

In the following, we can derive $u(z) = \dfrac{p(z)-1}{p(z)+1}$ (33)

$$(1-\theta)\left(1+\sum_{k=2}^{\infty}[k]_q \frac{a_{k-1}}{S} z^{k-1}\right) + \theta\left(1+\sum_{k=2}^{\infty}k[k]_q \frac{a_{k-1}}{S} z^{k-1}\right) + \sum_{k=2}^{\infty}\mu k(k-1)[k]_q \frac{a_{k-1}}{S} z^{k-1} =$$

$$1 + \left(\tfrac{1}{2}U_1(t)\right)c_1 z + \left(\frac{U_1(t)c_2}{2} - \frac{U_1(t)c_1{}^2}{4} + \frac{U_2(t)c_1{}^2}{4}\right)z^2$$

$$+\left(\frac{U_1(t)c_3}{2} - \frac{U_1(t)c_1\,c_2}{2} + \frac{U_1(t)c_1{}^3}{8} + \frac{U_2(t)c_1\,c_2}{2} - \frac{U_2(t)c_1{}^3}{4} + \frac{U_3(t)c_1{}^3}{8}\right)z^3 + \dots \tag{34}$$

Expanding and equating the coefficients of z, z^2 and z^3 in both sides of (34) gives:

$$(1+\theta+2\mu)[2]_q \frac{a_1}{S} = \tfrac{1}{2}U_1(t)c_1 \tag{35}$$

$$(1+2\theta+6\mu)[3]_q \frac{a_2}{S} = \frac{U_1(t)\,c_2}{2} - \frac{U_1(t)c_1{}^2}{4} + \frac{U_2(t)c_1{}^2}{4} \tag{36}$$

$$(1+3\theta+12\mu)[4]_q \frac{a_3}{S} = \frac{U_1(t)\,c_3}{2} - \frac{U_1(t)c_1\,c_2}{2} + \frac{U_1(t)c_1{}^3}{8}$$

$$+\frac{U_2(t)c_1\,c_2}{2} - \frac{U_2(t)c_1{}^3}{4} + \frac{U_3(t)c_1{}^3}{8} \tag{37}$$

which yields:

$$\frac{a_1}{S} = \frac{U_1(t)\,c_1}{2(1+\theta+2\mu)[2]_q} \tag{38}$$

$$\frac{a_2}{S} = \frac{U_1(t)\,c_2}{2(1+2\theta+6\mu)[3]_q} - \frac{U_1(t)\,c_1{}^2}{4(1+2\theta+6\mu)[3]_q} + \frac{U_2(t)\,c_1{}^2}{4(1+2\theta+6\mu)[3]_q} \tag{39}$$

$$\frac{a_3}{S} = \frac{U_1(t)\,c_3}{2(1+3\theta+12\mu)[4]_q} - \frac{U_1(t)\,c_1c_2}{2(1+3\theta+12\mu)[4]_q} + \frac{U_1(t)\,c_1{}^3}{8(1+3\theta+12\mu)[4]_q} \tag{40}$$

Applying lemmas 1.1 and 1.2 and (23) on (38), (39) and (40) together with triangle inequality we have:

$$\left|\frac{a_1}{S}\right| \le \frac{2t}{(1+\theta+2\mu)[2]_q},$$

$$\left|\frac{a_2}{S}\right| \leq \frac{2t^2-1}{(1+2\theta+6\mu)[3]_q}$$

$$\left|\frac{a_3}{S}\right| \leq \frac{4t(2t^2-1)}{(1+3\theta+12\mu)[4]_q}$$

which completes the proof.

Remarks: With special choices of parameters involved in theorem 1, various interesting results could be derived, these include:

Corollary 1.1 Let $f \in G_d(0, \mu, H(z,t))$. Then

$$\left|\frac{a_1}{S}\right| \leq \frac{2t}{(1+2\mu)[2]_q},$$

$$\left|\frac{a_2}{S}\right| \leq \frac{2t^2-1}{(1+6\mu)[3]_q}$$

$$\left|\frac{a_3}{S}\right| \leq \frac{4t(2t^2-1)}{(1+12\mu)[4]_q}$$

Corollary 1.2 Let $f \in G_d(1, \mu, H(z,t))$. Then

$$\left|\frac{a_1}{S}\right| \leq \frac{t}{(1+2\mu)[2]_q},$$

$$\left|\frac{a_2}{S}\right| \leq \frac{2t^2-1}{3(1+2\mu)[3]_q}$$

$$\left|\frac{a_3}{S}\right| \leq \frac{t(2t^2-1)}{(1+3\mu)[4]_q}$$

Corollary 1.3 Let $f \in G_d(\theta, \mu, H(z,\frac{1}{2}))$. Then

$$\left|\frac{a_1}{S}\right| \leq \frac{1}{M},$$

$$\left|\frac{a_2}{S}\right| \leq \frac{1}{2V}$$

$$\left|\frac{a_3}{S}\right| \leq \frac{1}{Y}$$

Corollary 1.4 Let $f \in G_d(\theta, 1, H(z,t))$. Then

$$\left|\frac{a_1}{S}\right| \leq \frac{2t}{(3+\theta)[2]_q},$$

$$\left|\frac{a_2}{S}\right| \leq \frac{2t^2-1}{(7+2\theta)[3]_q}$$

$$\left|\frac{a_3}{S}\right| \leq \frac{4t(2t^3-1)}{(13+3\theta)[4]_q}$$

where $M = (1+\theta+2\mu)[2]_q$, $V = (1+2\theta+6\mu)[3]_q$ and $Y = (1+3\theta+12\mu)[4]_q$

Theorem 2: Let $f \in H^t(\theta, \mu,)$, the Hankel determinant for the generalized distribution which satisfies the subordination principle. Then

$$\left|\frac{a_1}{S}\frac{a_3}{S} - \frac{a_2^2}{S^2}\right| \le \frac{4t}{MY} + \frac{2t(4t^2-3)}{Y} - \frac{(4t^2-1)^2}{V^2} \tag{41}$$

where $M = (1+\theta+2\mu)[2]_q$, $V = (1+2\theta+6\mu)[3]_q$ and $Y = (1+3\theta+12\mu)[4]_q$.

Proof:

Substituting for (27), (28), and (29) as obtained in Theorem 1 we have

$$\left|\frac{a_1}{S}\frac{a_3}{S} - \frac{a_2^2}{S^2}\right| = \frac{U_1(t)c_1}{2(1+\theta+2\mu)[2]_q}\left[\frac{U_1(t)c_1}{2(1+3\theta+12\mu)[4]_q} - \frac{U_1(t)c_1c_2}{2(1+3\theta+12\mu)[4]_q} + \frac{U_1(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q}\right.$$
$$\left. + \frac{U_2(t)c_1c_2}{2(1+3\theta+12\mu)[4]_q} - \frac{U_2(t)c_1^{\,3}}{4(1+3\theta+12\mu)[4]_q} + \frac{U_3(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q}\right]$$
$$-\left[\frac{U_1(t)c_2}{2(1+2\theta+6\mu)[3]_q} - \frac{U_1(t)c_1^{\,2}}{4(1+2\theta+6\mu)[3]_q} + \frac{U_2(t)c_1^{\,2}}{4(1+2\theta+6\mu)[3]_q}\right]^2 \tag{42}$$

From lemma 1.2 $2c_2 = c_1^{\,2} + x(4-c_1^{\,2})$ and

$4c_3 = c_1^{\,3} + 2(4-c_1^{\,2})c_1^{\,x} - c_1(4-c_1^{\,2})x^2 + 2(4-c_1^{\,2})(1-|x|^2)z$ for some $z, |z| \le 1$

Thus

$$\left|\frac{a_1}{S}\frac{a_3}{S} - \frac{a_2^2}{S^2}\right| = \frac{U_1^{\,2}(t)c_1^{\,4}}{16(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q} + \frac{U_1^{\,2}(t)(4-c_1^{\,2})c_1^{\,2}x}{8(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q}$$
$$-\frac{U_1^{\,2}(t)(4-c_1^{\,2})c_1^{\,2}x^2}{16(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q} + \frac{U_1^{\,2}(t)(4-c_1^{\,2})(1-|x|^2)c_1z}{8(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q}$$
$$-\frac{U_1(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q} - \frac{U_1(t)(4-c_1^{\,2})c_1x}{4(1+3\theta+12\mu)[4]_q} + \frac{U_2(t)(4-c_1^{\,2})c_1x}{4(1+3\theta+12\mu)[4]_q} + \frac{U_3(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q}$$
$$-\frac{U_1^{\,2}(t)(4-c_1^{\,2})^2x^2}{4(1+3\theta+12\mu)[4]_q} - \frac{U_2^{\,2}(t)c_1^{\,4}}{16(1+2\theta+6\mu)^2[3]_q^{\,2}} - \frac{U_1(t)U_2(t)c_1^{\,2}(4-c_1^{\,2})x}{8(1+2\theta+6\mu)^2[3]_q^{\,2}} \tag{43}$$

Let R = $4-c_1^{\,2}$ and W = $(1-|x|^2)z$. Then (43) becomes

$$\left|\frac{a_1}{S}\frac{a_3}{S} - \frac{a_2^2}{S^2}\right| = \frac{U_1^{\,2}(t)c_1^{\,4}}{16(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q} + \frac{U_1^{\,2}(t)Rc_1^{\,2}x}{8(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q}$$
$$-\frac{U_1^{\,2}(t)Rc_1^{\,2}x^2}{16(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q} + \frac{U_1^{\,2}(t)Rwc_1}{8(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q} - \frac{U_1(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q}$$
$$-\frac{U_1(t)Rc_1x}{4(1+3\theta+12\mu)[4]_q} + \frac{U_2(t)Rc_1x}{4(1+3\theta+12\mu)[4]_q} + \frac{U_3(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q}$$
$$-\frac{U_1^{\,2}(t)R^2x^2}{16(1+2\theta+6\mu)^2[3]_q^{\,2}} - \frac{U_2^{\,2}(t)c_1^{\,4}}{16(1+2\theta+6\mu)^2[3]_q^{\,2}} - \frac{U_1(t)U_2(t)c_1^{\,2}xR}{8(1+2\theta+6\mu)^2[3]_q^{\,2}} \tag{44}$$

By Lemma 1.1, we have $|c_n| \le 2,\ n \in \mathbb{N} = \{1,2,3,\ldots\}$. For convenience of notation we take $c_n = c$ and we may assume without loss of generality that $c \in [0,2]$. Applying the triangle inequality with $R = 4-c_1^{\,2}$.

Then $\left|\frac{a_1}{S}\frac{a_3}{S}-\frac{a_2^2}{S^2}\right| \le \left|(\lambda_1-\lambda_2)c^4+(\lambda_3-\lambda_4)c^3\right|+\frac{U_1^{\,2}(t)\,Rc^2x}{8(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q}$

$$-\frac{U_1^{\,2}(t)\,Rc^2x^2}{16(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q}+\frac{U_1^{\,2}(t)\,Rwc}{8(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q}$$

$$-\frac{U_1(t)\,Rcx}{4(1+3\theta+12\mu)[4]_q}+\frac{U_2(t)\,Rcx}{4(1+3\theta+12\mu)[4]_q}-\frac{U_1^{\,2}(t)\,R^2x^2}{16(1+2\theta+6\mu)^2[3]_q^{\,2}}$$

$$-\frac{U_1(t)U_2(t)\,c^2Rx}{8(1+2\theta+6\mu)^2[3]_q^{\,2}}=\phi(|x|) \qquad (45)$$

Trivially, we then show that this expression has a maximum value

$$\left|\frac{a_1}{S}\frac{a_3}{S}-\frac{a_2^2}{S^2}\right| \le \frac{U_1^{\,2}(t)}{(1+\theta+2\mu)(1+3\theta+12\mu)[2]_q[4]_q}-\frac{U_1(t)}{(1+3\theta+12\mu)[4]_q}$$

$$+\frac{U_3(t)}{(1+3\theta+12\mu)[4]_q}-\frac{U_2^{\,2}(t)}{(1+2\theta+6\mu)^2[3]_q^{\,2}} \qquad (46)$$

on [0, 2] when c = 2

Applying (23) on (46) gives the result

$$\left|\frac{a_1}{S}\frac{a_3}{S}-\frac{a_2^2}{S^2}\right| \le \frac{4t}{MY}+\frac{2t(4t^2-3)}{Y}-\frac{(4t^2-1)^2}{V^2}$$

where $M=(1+\theta+2\mu)[2]_q$, $V=(1+2\theta+6\mu)[3]_q$ and $Y=(1+3\theta+12\mu)[4]_q$.

which completes the proof.

Corollary 2.1: Let $f \in H^t(\theta,\mu)$ in theorem 2. Then

$$\left|\frac{a_1}{S}\frac{a_3}{S}-\frac{a_2^2}{S^2}\right| \le \frac{4t}{(1+2\mu)(1+12\mu)[2]_q[4]_q}+\frac{2t(4t^2-3)}{(1+2\mu)[4]_q}+\left(\frac{4t^2-1}{(1+6\mu)[3]_q}\right)^2$$

Corollary 2.2 Let $f \in H^t(1,\mu)$ in theorem 2. Then

$$\left|\frac{a_1}{S}\frac{a_3}{S}-\frac{a_2^2}{S^2}\right| \le \frac{2t}{4(1+\mu)(1+3\mu)[2]_q[4]_q}+\frac{t(4t^2-3)}{2(1+3\mu)[4]_q}+\left(\frac{4t^2-1}{3(1+2\mu)[3]_q}\right)^2$$

Corollary 2.3 Let $f \in H^t(\theta,1)$ in theorem 2. Then

$$\left|\frac{a_1}{S}\frac{a_3}{S}-\frac{a_2^2}{S^2}\right| \le \frac{4t}{(3+\theta)(13+3\theta)[2]_q[4]_q}+\frac{2t(4t^2-3)}{(13+3\theta)[4]_q}-\left(\frac{4t^2-1}{(7+2\theta)[3]_q}\right)^2$$

Corollary 2.4 Let $f \in H^{\frac{1}{2}}(\theta,\mu)$ in theorem 2. Then

$$\left|\frac{a_1}{S}\frac{a_3}{S}-\frac{a_2^2}{S^2}\right| \le \frac{2}{MY}-\frac{2}{Y}$$

Theorem 3. Let $f \in H^t(\theta,\mu)$, the Hankel determinant for the generalized distribution which satisfies the subordination principle. Then

$$\left|\frac{a_1}{S}\frac{a_2}{S}-\frac{a_3}{S}\right| \le \frac{2t\,(4t^2-1)}{MV}-\frac{4t\,(2t^2-1)}{Y} \qquad (47)$$

where $M=(1+\theta+2\mu)[2]_q$, $V=(1+2\theta+6\mu)[3]_q$ and $Y=(1+3\theta+12\mu)[4]_q$.

Proof: Substituting the values of $\frac{a_1}{S}$, $\frac{a_2}{S}$, and $\frac{a_3}{S}$ as in (27), (28) and (29) gives

$$\left|\frac{a_1}{S}\frac{a_2}{S}-\frac{a_3}{S}\right|=\frac{U_1(t)c_1}{2(1+\theta+2\mu)[2]_q}\left[\frac{U_1(t)c_2}{2(1+2\theta+6\mu)[3]_q}-\frac{U_1(t)c_1^{\,2}}{4(1+2\theta+6\mu)[3]_q}+\frac{U_2(t)c_1^{\,2}}{4(1+2\theta+6\mu)[3]_q}\right]$$
$$-\frac{U_1(t)c_3}{2(1+3\theta+12\mu)[4]_q}+\frac{U_1(t)c_1c_2}{2(1+3\theta+12\mu)[4]_q}-\frac{U_1(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q}-\frac{U_2(t)c_1c_2}{2(1+3\theta+12\mu)[4]_q}$$
$$+\frac{U_2(t)c_1^{\,3}}{4(1+3\theta+12\mu)[4]_q}-\frac{U_3(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q} \tag{48}$$

Applying lemma 1.2, we have

$$\left|\frac{a_1}{S}\frac{a_2}{S}-\frac{a_3}{S}\right|=\frac{U_1^{\,2}(t)\left(4-c_1^{\,2}\right)c_1x}{16(1+\theta+2\mu)(1+2\theta+6\mu)[2]_q[3]_q}+\frac{U_1(t)U_2(t)c_1^{\,3}}{8(1+\theta+2\mu)(1+2\theta+6\mu)[2]_q[3]_q}$$
$$-\frac{U_1(t)\left(4-c_1^{\,2}\right)c_1x}{4(1+3\theta+12\mu)[4]_q}-\frac{U_1(t)\left(4-c_1^{\,2}\right)c_1x^2}{8(1+3\theta+12\mu)[4]_q}-\frac{U_1(t)\left(4-c_1^{\,2}\right)\left(1-|x|^2\right)z}{4(1+3\theta+12\mu)[4]_q}$$
$$-\frac{U_2(t)\left(4-c_1^{\,2}\right)c_1x}{8(1+3\theta+12\mu)[4]_q}-\frac{U_3(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q} \tag{49}$$

Let R = $4-c_1^{\,2}$ and W = $\left(1-|x|^2\right)z$ in (49). Then,

$$\left|\frac{a_1}{S}\frac{a_2}{S}-\frac{a_3}{S}\right|=\frac{U_1^{\,2}(t)Rc_1x}{16(1+\theta+2\mu)(1+2\theta+6\mu)[2]_q[3]_q}+\frac{U_1(t)U_2(t)c_1^{\,3}}{8(1+\theta+2\mu)(1+2\theta+6\mu)[2]_q[3]_q}$$
$$-\frac{U_1(t)Rc_1x}{4(1+3\theta+12\mu)[4]_q}-\frac{U_1(t)Rc_1x^2}{8(1+3\theta+12\mu)[4]_q}-\frac{U_1(t)Rw}{4(1+3\theta+12\mu)[4]_q}$$
$$-\frac{U_2(t)Rc_1x}{8(1+3\theta+12\mu)[4]_q}-\frac{U_3(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q} \tag{50}$$

By Lemma 1.1, we have $|c_n|\le 2,\ n\in\mathbb{N}=\{1,2,3,...\}$. For convenience of notation we take $c_n=c$ and we may assume without loss of generality that $c\in[0,2]$. Applying the triangle inequality with $R=4-c_1^{\,2}$. Then

$$\left|\frac{a_1}{S}\frac{a_2}{S}-\frac{a_3}{S}\right|\le\frac{U_1(t)U_2(t)c^3}{8(1+\theta+2\mu)(1+2\theta+6\mu)[2]_q[3]_q}-\frac{U_1(t)c^3}{8(1+3\theta+12\mu)[4]_q}$$
$$+\frac{U_1^{\,2}(t)Rcx}{16(1+\theta+2\mu)(1+2\theta+6\mu)[2]_q[3]_q}-\frac{U_1(t)Rcx}{4(1+3\theta+12\mu)[4]_q}+\frac{U_1(t)Rcx^2}{8(1+3\theta+12\mu)[4]_q}$$
$$-\frac{U_1(t)Rw}{4(1+3\theta+12\mu)[4]_q}-\frac{U_2(t)Rcx}{8(1+3\theta+12\mu)[4]_q}=\phi(|x|) \tag{51}$$

We can then show that this expression has a maximum value

$$\left|\frac{a_1}{S}\frac{a_2}{S}-\frac{a_3}{S}\right|\le\frac{U_1(t)U_2(t)}{(1+\theta+2\mu)(1+2\theta+6\mu)[2]_q[3]_q}-\frac{U_3(t)}{(1+3\theta+12\mu)[4]_q} \tag{52}$$

on $[0,2]$ when c =2

Applying (23) on (52) yields the expected result.

Thus $\left|\frac{a_1}{S}\frac{a_2}{S}-\frac{a_3}{S}\right| \le \frac{2t\left(4t^2-1\right)}{MV}-\frac{4t\left(2t^2-1\right)}{Y}$

where $M=(1+\theta+2\mu)[2]_q$, $V=(1+2\theta+6\mu)[3]_q$ and $Y=(1+3\theta+12\mu)[4]_q$.

Thus the proof is completed.

Remarks: With special choices of parameters involved in theorem 1, various interesting results could be derived.

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