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Stability Criteria of Impulsive Differential Equations byLyapunov's Second Method

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Abstract:*During the history of the earth different kinds of processes by which living organisms are believed to have developed from earlier forms experience a change of state abruptly at certain moments of time. These processes act like short term perturbations and its duration is negligible in comparison with the total duration of the process. That is differential equations with impulse effects called impulsive differential equations. In recent years there have been intensive studies on the behavior of the solution of impulsive differential equations. It has wide applications in physics, population dynamics, ecology, biology, industrial robotics, biotechnology, optimal control theory etc. The Lyapunovs second method is used as a tool to obtain the stability of impulsive differential equations.*

Key Word*: impulsive differential equations, Lyapunov's second method, asymptotic stability*

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I. Introduction

In recent years the theory of impulsive differential equations have had a great development. In this paper we discuss the basic definitions of impulsive differential equations and by employing the Lyapunov's second method we investigate the stability and asymptotic stability of the zero solution of the impulsive differential equations. Consider the impulsive differential equations of the form

$$\frac{dx}{dt} = f(t,x), \qquad t \neq \theta_i(x),$$
$$\Delta x(t) = J_i(x), \qquad t = \theta_i(x), i \in N = \{1, 2, \dots \dots \dots .\} \tag{1}$$

where $t \in [a, \infty), x \in R^n$, $f \in C^{p+1}([a,\infty) \times D;\ R^n)$, $\Delta x(t) = x(\theta^+) - x(\theta);\ where\ t = \theta$, and $\theta_i(x), i \in N = \{1, 2, \dots \dots \dots .\}$ is a sequence of impulse action times. For our purpose we introduce the notation

$$G = \{(t,x): t \geq 0, x \in S_p\}, \qquad S_p = \{x \in R^n: \|x\| < \rho\},$$

where$\rho > 0$ is a fixed real number and $\|x\|$ denotes Euclidean norm of $x \in R^n$. Next for each $i \in N$ we define $\theta_i^0 = \theta_i(0)$ and let
$G_i = \{(t,x) \in G: t$is between θ_i^0 and $\theta_i(x)\}$.

Definition 1.1: The zero solution of (1) is called *stable* if for any given $\varepsilon > 0$ and $t_0 \in R^+$ there exists a $\delta = \delta(\varepsilon, t_0)$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$.

Definition 1.2: The zero solution of (1) is called *asymptotically* stable if it is stable and there exists a$\bar{\delta} > 0$ such that for any $x(t, t_0, x_0)$ with $\|x_0\| < \bar{\delta}$satisfies $\lim_{t\to\infty} x(t) = 0$

Definition 1.3: A function *V(t,x)* is called positive definite on G if there exists a$\in R$ such that*V(t, x*) such that *V(t, x*) ≥ a($\|x\|$) for all (t, x) ∈ G; it is called positive semi definite on G if *V(t,x)*≥ 0 for all (t, x) ∈ G. The function *V(t,x)* is called negative definite (negative semi definite) on G if $-$*V(t,x)* is positive definite (positive semi definite) on G.

II. Lyapunov's Second Method of stability

Theorem 2.1: Assume that the following conditions are fulfilled,

(i) $V(t,x)$is positive definite on G.
(ii) $D^{+}V(t,x)$is negative semidefinite on G.
(iii) $D^{+}V(t,x) \leq -\varphi\,(V(t,x))$ for some $\varphi \in A$ and for all $(t,x) \in \bigcup_{i\in N} G_i$
(iv) $V(\theta_i(x), x + J_i(x)) \leq \mu(V(\theta_i(x),x))$ for some $\mu \in A$ and for all x$\in S_p$and i$\in$N.
(v) There exists an L > 0 such that $|\theta_i(x) - \theta_i(y)| \leq L\|x - y\|$ for all x, y $\in S_p$ and i$\in N$.
(vi) There exists an L_1> 0 such that $\left|\theta_i(x) - \theta_i^0\right| \geq L_1\|x\|$ for all x $\in S_p$ and i$\in N$.
(vii) There exists a $\gamma \geq 0$ such that

$\int_{V(\theta_i(x),x)}^{\mu(V(\theta_i(x),x))} \frac{ds}{\mu(s)} \leq (L_1 - \gamma)$for all x $\in S_p$ and i $\in N$.

Proof: This theorem is proved in the paper [6] and the references cited there in. In this paper we work with two different systems of impulsive differential equations to prove the stability of the impulsive differential equation.

III. Examples And Analysis

Example 3.1: Let us consider the following impulsive system

$$\dot{x_1} = \begin{cases} -x_1, & (t,x) \notin S \\ -x_2, & (t,x) \in S \end{cases}$$

$$\dot{x_2} = \begin{cases} -x_2, & (t,x) \notin S \\ x_1, & (t,x) \in S \end{cases}$$

$\Delta x_1 = -\alpha x_1 + \beta x_2$,$\Delta x_2 = \beta x_1 - \alpha x_2$

We choose $V(x)$= $x_1^2 + x_2^2$ and make the following observations

(a) $\dot{V}(x) = -2V(x)$if$(t,x) \notin$ S and $\dot{V}(x) = 0 (t,x) \in S$. Since $V(x + \Delta x)$=$((1-\alpha)^2 + \beta^2)\,(x_1^2 + x_2^2) - 4\beta(\alpha - 1)x_1x_2$

where $l(\alpha,\beta)$=$(|\beta| + |1-\alpha|)^2$

(b) $\|x + \Delta x\|^2$ =$((1-\alpha)^2 + \beta^2)\,(x_1^2 + x_2^2) - 4\beta(\alpha - 1)x_1x_2$ and $\|x + \Delta x\|^2 \geq (|1-\alpha| - |\beta|)^2\|x\|^2$ and it follows that $\left||1-\alpha| - |\beta|\right| \geq 1$, then

$$\theta_i(x + \Delta x) \leq \theta_i(x).$$

(c) $\left|\theta_i(x) - \theta_i^0\right| = \sqrt{x_1^2 + x_2^2} = \|x\|$
(d) Let $\varphi(s) = 2s$ *and* $\mu(s) = l(\alpha,\beta)s$ *and* fix a positive number $\gamma < 1$ and if $l(\alpha,\beta) \leq 1$, then $l(\alpha,\beta) \leq 2(1-\gamma)\|x\|$ and hence

$$\int_V^{lV} \frac{ds}{2s} \leq (1-\gamma)\|x\|$$

Now from the theorem (2.1) we can state that the zero solution of the given system of impulsive differential equations is asymptotically stable if

$\left||1-\alpha| - |\beta|\right| \geq 1and\left||1-\alpha| + |\beta|\right| \leq 1$

Example 3.2: Let$\theta_i(x) = i - \sqrt{x_1^2 + x_2^2}$ so that $G_i = \{(t,x) \in G{:}\, i - \sqrt{x_1^2 + x_2^2} < 1 \leq t\}$. We define $S = \bigcup_{i=1}^{\infty} G_i$ and consider the impulsive differential equations

$$\frac{dx_1}{dt} = -x_1^3 + x_1x_2^2, \qquad \frac{dx_2}{dt} = -x_1^2x_2, \qquad t \notin S$$

$\Delta x_1 = -\alpha x_1 + \beta x_2,$
$\Delta x_2 = \beta x_1 - \alpha x_2$

We choose *V(x)*= $(x_1^2 + x_2^2)/2$ and make the following observations

$(a)\dot{V}(x) = -x_1^4 < V(t,x)if(t,x) \notin$ S and $\dot{V}(x) = 0(t,x) \in S.$

Since *V(x + Δx)*=$((1-\alpha)^2 + \beta^2)$ $(x_1^2 + x_2^2)$ $-4\beta(\alpha - 1)x_1x_2$
where $l(\alpha,\beta)$= $(|\beta| + |1-\alpha|)^2$
$V(t,x)$is positive definite on G and $D^+V(t,x)$ is negative semidefinite on G.

$(b)\|x + \Delta x\|^2$ =$((1-\alpha)^2 + \beta^2)$ $(x_1^2 + x_2^2)$ $-4\beta(\alpha-1)x_1x_2$ and $\|x + \Delta x\|^2 \geq (|1-\alpha| - |\beta|)^2\|x\|^2$
and it follows that $\big||1-\alpha| - |\beta|\big| \geq 1$, then
$\theta_i(x + \Delta x) \leq \theta_i(x).$
$(c)\left|\theta_i(x) - \theta_i^0\right| = \sqrt{x_1^2 + x_2^2} = \|x\|$
(d) Let $\varphi(s) = 2s$ *and* $\mu(s) = l(\alpha,\beta)s$ *and* fix a positive number $\gamma < 1$ and if $l(\alpha,\beta) \leq 1,$
then$l(\alpha,\beta) \leq (1-\gamma)\|x\|$ and hence

$$\int_V^{lV} \frac{ds}{s} \leq (1-\gamma)\|x\|$$

Now from the theorem (2.1) we can state that the zero solution of the given system of impulsive differential equations is asymptotically stable if

$\big||1-\alpha| - |\beta|\big| \geq 1and\big||1-\alpha| + |\beta|\big| \leq 1$

It is easy to verify that all the conditions are satisfied according to theorem (2.1) except condition (v). Now if we consider that the family $\{\theta_i^x\}$ is equicontinuous at $x = 0$ and that $\theta_i^0 \geq \theta_i(x)$ for all $x \in S_p$**.** Then the theorem (2.1) is valid and the given impulsive differential equations are both stable and asymptotically stable.

IV. Conclusion

By using Lyapunov's direct method as a tool we solved two different systems of equations with impulse effects in this paper and discussed their stability criteria. In my paper I intend to work with generalized conditions for Lyapunov's Second method with proper examples and stronger results.

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