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Transformation of Non-NDD Function to NDD Function

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Abstract - *A function T(s) whose numerator polynomial is the derivative of the denominator polynomial is designated as NDD function. It is shown that a class of non-NDD (NNDD) functions (a ratio of finite length polynomials) can be converted into an NDD function by multiplying both the numerator and denominator by a suitable function. Expression for this function is derived. Transformation procedure is developed and illustrated with several examples. Its application in obtaining partial fraction expansion is given and compared with the routine method. How an NDD function can be transformed into an NNDD function is mentioned. Finally a property of NDD function is stated and proved.*

Keywords *- NDD functions, Partial fraction expansion, RPPFC Theorem, PFRPC theorem*

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I. INTRODUCTION

A function $T(s)$ whose numerator polynomial N(s) is the derivative of the denominator polynomial D(s) is designated as NDD function [1]. In many applications, one has to do the partial fraction expansion which requires lengthy algebra. However, if $T(s)$ is an NDD function, the algebra can be completely eliminated using Ratio of Polynomials to the Partial Fractions Conversion (RPPFC) theorem [1]. One can get back the original function with or without the common factors present employing the Partial Fraction to Ratio of Polynomials Conversion (PFRPC) theorem.

It is shown that a Non-NDD (NNDD) function $T(s)$ can be transformed into an NDD function $T_1(s)$ by simultaneously multiplying both the numerator and denominator of the NNDD function by a suitable function. Before doing so, the properties of the NDD functions [1] are summarized in Section II for ready reference. Section III deals with the conversion of NNDD to NDD functions. Section IV gives illustrative examples. Section V gives the application of the transformation in obtaining the partial fraction expansion. Section VI mentions how the NDD function can be converted into NNDD function. Lastly, Section VII derives a property of NDD functions followed by conclusion in Section VIII.

II. PROPERTIES OF NDD FUNCTIONS

1. There may be common factors present in the numerator and denominator of an NDD function. After cancelling any of the common factors, the remaining function does not remain an NDD function.
2. derived. Transformation procedure has been developed and illustrated with several examples. Its denominator by a suitable function. Expression for this function has been derived. Transformation procedure has been developed and illustrated with several examples. Its application in obtaining partial fraction expansion has been given. How an NDD function can be transformed into an NNDD function The highest power of the numerator polynomial is at least one less than that of the denominator polynomial. Hence, it has at least one zero at infinity.
3. The lowest power of the numerator polynomial is at least one more than or equal to that of the denominator polynomial. Hence, an NDD function has neither multiple poles nor zeros at the origin.
4. There are two possible denominator polynomials, with or without a constant term, for which there will be only one numerator polynomial.
5. Sum of two NDDs is also an NDD.
6. After removing completely any number of poles of multiplicity m_i the remaining function is also an NDD function. This is like driving point function where after removing a pole the remaining function is again driving point function.
7. Driving point functions of RC, RL and LC networks are NDD functions.
8. Ratio of Polynomials to the Partial Fractions Conversion (RPPFC) theorem:

If F(s) is an NDD function such that it has k number of poles and ith pole is of multiplicity mi then it can be expressed as

$$f(s) = \frac{\frac{d}{ds}\prod_{i=1}^{k}(s+p_i)^{m_i}}{\prod_{i=1}^{k}(s+p_i)^{m_i}} = \sum_{i=1}^{k}\frac{m_i}{(s+p_i)}$$

III. Transformation of an Non-NDD function into an NDD function

Theorem: A Non-NDD function $T(s) = \frac{N(s)}{D(s)}$ can be transformed into an NDD function T1(s) by multiplying $N(s)$ and $D(s)$ by a suitable function

Proof: Let us multiply both the numerator and denominator by a function $H(s)$. Then

$$T(s) = \frac{N(s)}{D(s)} = \frac{N(s)H(s)}{D(s)H(s)} = \frac{N_1(s)}{D_1(s)}$$
$$= T_1(s)$$

If $T_1(s)$ is to represent an NDD function then

$$N_1(s) = \frac{d}{ds}D_1(s) = \frac{d}{ds}D(s)H(s)$$
$$N(s)H(s) = H(s)\frac{d}{ds}D(s) + D(s)\frac{d}{ds}H(s)$$
$$\Rightarrow \frac{\frac{d}{ds}H(s)}{H(s)} = T(s) - \frac{\frac{d}{ds}D(s)}{D(s)}$$

Integrating both sides, we get

$$\ln H(s) = \int T(s)ds - \ln D(s) + \ln A$$

where A is the constant of integration. Taking antilog of both sides, we get

$$H(s) = \frac{Ae^{\int T(s)ds}}{D(s)} \tag{1}$$

Thus, the resultant NDD function is

$$T_1(s) = \frac{N_1(s)}{D_1(s)} = \frac{N(s)H(s)}{D(s)H(s)} = \frac{N(s)e^{\int T(s)ds}/D(s)}{e^{\int T(s)ds}/D(s)} \tag{2}$$

Caution: One should not attempt to cancel the common factor(s) in N(s)H(s) and D(s)H(s); otherwise it will result in the same NNDD function.

If T(s) itself is an NDD function, i.e.,

$$N(s) = \frac{d}{ds}D(s)$$

then from (1), $H(s) = 1$. Hence, from (2)

$$T_1(s) = T(s).$$

IV. Some Illustrative Examples

Example 1: Transform

$$T(s) = \frac{1}{(s+\sigma)} \tag{3}$$

into an NDD function.

Since, $T(s)$ is already an NDD function, from (2), $T_1(s) = \dfrac{1}{(s+\sigma)}$ is the required NDD function.

Example 2: Transform

$$T(s) = \frac{1}{(s+\alpha)(s+\beta)}. \tag{4}$$

into NDD function

Case (i) $\beta \neq \propto$

Since,

$$\int T(s)ds = \int \frac{ds}{(s+\alpha)(s+\beta)} = \int \left(\frac{1}{\beta-\alpha}\right)\left(\frac{1}{(s+\alpha)} - \frac{1}{(s+\beta)}\right)ds$$

$$= \ln\left(\frac{s+\alpha}{s+\beta}\right)^{\left(\frac{1}{\beta-\alpha}\right)}.$$

From (2),

$$T_1(s) = \frac{\frac{1}{(s+\alpha)(s+\beta)}\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}}{\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}} \tag{5}$$

Case (ii) When $\beta = \propto$

When $\beta \to \propto$, (5) reduces to an inderminant form. To overcome this difficulty, we proceed as follows.

$$\lim_{\alpha\to\beta} T_1(s) = \lim_{\alpha\to\beta} \frac{\frac{1}{(s+\propto)(s+\beta)}\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}}{\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}}$$

Taking log on both sides, we get

$$\ln\left\{\lim_{\alpha\to\beta} T_1(s)\right\} = \ln\left\{\lim_{\alpha\to\beta} \frac{\frac{1}{(s+\propto)(s+\beta)}\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}}{\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}}\right\}$$

Interchanging the operation of limit and log, we get

$$\lim_{\alpha\to\beta}\{\ln T_1(s)\} = \lim_{\alpha\to\beta}\left[\ln\frac{1}{(s+\propto)(s+\beta)} + \ln\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}} - \ln\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}\right]$$

$$= \left[\lim_{\alpha\to\beta}\left\{\ln\left(\frac{1}{(s+\propto)(s+\beta)}\right)\right\} + \lim_{\alpha\to\beta}\left\{\ln\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}\right\}\right] - \lim_{\alpha\to\beta}\left\{\ln\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}\right\}$$

$$= \left[\ln\left\{\lim_{\alpha\to\beta}\frac{1}{(s+\propto)(s+\beta)}\right\} + \lim_{\alpha\to\beta}\left\{\ln\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}\right\}\right] - \lim_{\alpha\to\beta}\left\{\ln\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}\right\}$$

$$= \left[\ln\frac{1}{(s+\beta)^2} + \lim_{\alpha\to\beta}\left\{\ln\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}\right\}\right] - \lim_{\alpha\to\beta}\left\{\ln\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}\right\} \tag{6}$$

Now ,

$$\lim_{\alpha\to\beta}\left\{\ln\left(\frac{s+\alpha}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}\right\} = \lim_{\alpha\to\beta}\left\{\ln\left(1+\frac{\alpha-\beta}{s+\beta}\right)^{\frac{1}{\beta-\alpha}}\right\} = \lim_{\alpha\to\beta}\left\{\left(\frac{1}{\beta-\alpha}\right)\left(\frac{\alpha-\beta}{s+\beta}\right)\right\}$$

$$= \lim_{\alpha\to\beta}\left[\left(\frac{1}{\beta-\alpha}\right)\left\{\left(\frac{\alpha-\beta}{s+\beta}\right)-\frac{1}{2}\left(\frac{\alpha-\beta}{s+\beta}\right)^2+\frac{1}{3}\left(\frac{\alpha-\beta}{s+\beta}\right)^3-\frac{1}{4}\left(\frac{\alpha-\beta}{s+\beta}\right)^4\dots.\right\}\right]$$

$$= \lim_{\alpha\to\beta}\left[\left\{\left(\frac{-1}{s+\beta}\right)+\frac{1}{2}\left(\frac{\alpha-\beta}{s+\beta}\right)-\frac{1}{3}\left(\frac{\alpha-\beta}{s+\beta}\right)^2+\frac{1}{4}\left(\frac{\alpha-\beta}{s+\beta}\right)^3\dots.\right\}\right]$$

$$= \frac{-1}{s+\beta} \tag{7}$$

From Eq. (6), we get

$$\ln\left\{\lim_{\alpha\to\beta}T_1(s)\right\} = \left[\ln\frac{1}{(s+\beta)^2}+\ln e^{\frac{-1}{s+\beta}}\right]-\ln e^{\frac{-1}{s+\beta}} = \ln\frac{\frac{1}{(s+\beta)^2}e^{\frac{-1}{s+\beta}}}{e^{\frac{-1}{s+\beta}}}$$

$$\rightarrow \lim_{\alpha\to\beta}T_1(s) = \frac{\frac{1}{(s+\beta)^2}e^{\frac{-1}{s+\beta}}}{e^{\frac{-1}{s+\beta}}}$$

The same result can be obtained very easily as follows. Note that

$$\int\frac{1}{(s+\beta)^2}ds = -\frac{1}{s+\beta}.$$

Using (2), we get

$$T_1(s) = \frac{\frac{e^{-\frac{1}{s+\beta}}}{(s+\beta)^2}}{e^{-\frac{1}{s+\beta}}}.$$

Thus a single multiple pole in $T(s)$ will not result in $T_1(s)$ as a ratio of rational polynomials.

Example 3: Transform

$$T(s) = \frac{1}{s^2+\mu^2} = \frac{1}{(s+j\mu)(s-j\mu)} \tag{8}$$

Here $\alpha = j\mu,\ \ \beta = -j\mu$.

Using equation (5)

$$T_1(s) = \frac{\frac{1}{(s^2+\mu^2)}\left(\frac{s-j\mu}{s+j\mu}\right)^{\frac{1}{j2\mu}}}{\left(\frac{s-j\mu}{s+j\mu}\right)^{\frac{1}{j2\mu}}} \tag{9}$$

Alternative approach

Since $D(s) = s^2+\mu^2$

$$\int T(s)ds = \frac{1}{\mu}\tan^{-1}\left(\frac{s}{\mu}\right) \tag{10}$$

Example 4: Transform

$$T(s) = \cot(s) = \frac{\cos s}{\sin s}$$

is already an NDD function. Therefore, $T_1(s) = \dfrac{\cos(s)}{\sin(s)}$ is the required NDD function.

Example 5: Transform

$$T(s)=\frac{1}{\left(s^2+s+1\right)} \tag{11}$$

Here $\alpha=0.5+j0.5\sqrt{3}$, $\beta=.\ 0.5-j0.5\sqrt{3}$. From (5)

$$T_1(s)=\frac{\frac{1}{\left(s^2+s+1\right)}\left(\frac{s+0.5-j0.5\sqrt{3}}{s+0.5+j0.5\sqrt{3}}\right)^{\frac{1}{j\sqrt{3}}}}{\left(\frac{s+0.5-j0.5\sqrt{3}}{s+0.5+j0.5\sqrt{3}}\right)^{\frac{1}{j\sqrt{3}}}} \tag{12}$$

V. APPLICATION IN PARTIAL FRACTION EXPANSION

Partial fraction expansion is possible for function whose denominator polynomial can be factored. If the NDD function is given by (9), then by applying RPPFC theorem, we can immediately express it in terms of partial fractions as

$$T(s)=\left(\frac{1}{j\sqrt{3}}\right)\frac{1}{s+0.5-j0.5\sqrt{3}}-\left(\frac{1}{j\sqrt{3}}\right)\frac{1}{s+0.5+j0.5\sqrt{3}}$$

Comparison with Conventional Method

Example 6: Let

$$T(s)=\frac{10s^4+28s^3-80s^2-176s+18}{s^5+3s^4-15s^3-35s^2+54s+72}=\frac{10s^4+28s^3-80s^2-176s+18}{(s-3)(s-2)(s+1)(s+3)(s+4)} \tag{13}$$

Conventional method:

$$\frac{10s^4+28s^3-80s^2-176s+18}{(s-3)(s-2)(s+1)(s+3)(s+4)}=\frac{A}{(s-3)}$$
$$+\frac{B}{(s-2)}+\frac{C}{(s+1)}+\frac{D}{(s+3)}+\frac{E}{(s+4)}$$

Putting s = 3, we get

$$A=2$$

Putting s = 2

$$B=\frac{(160+224-320-352+18)}{(-1)(3)(5)(6)}=3$$

Putting $s=-1$,

$$C=\frac{(10-28-80+176+18)}{(-4)(-3)(2)(3)}=\frac{4}{3}$$

Putting $s=-3$,

$$D=\frac{(810-756-720+528+18)}{(-6)(-5)(-2)(1)}=2$$

Putting s = -4,

$$E=\frac{(2560-1792-1280+704+18)}{(-7)(-6)(-3)(-1)}=\frac{5}{3}$$

Thus

$$T(s)=\frac{2}{(s-3)}+\frac{3}{(s-2)}+\frac{4}{3(s+1)}+\frac{2}{(s+3)}+\frac{5}{3(s+4)} \tag{14}$$

Method based on RPPFC Theorem:

$$T(s)=\frac{2\left(5s^4+12s^3-45s^2-70s+54\right)}{s^5+3s^4-15s^3-35s^2+54s+72}+\frac{\left(s^2-9\right)(4s+10)}{(s-3)(s-2)(s+1)(s+3)(s+4)} \tag{15}$$

$$= F_1(s) + F_2(s).$$

where $F_1(s)$ is an NDD function and $F_2(s)$ is an NNDD. Using RPPFC theorem

$$F_1(s) = \frac{2}{(s-3)} + \frac{2}{(s-2)} + \frac{2}{(s+1)} + \frac{2}{(s+3)} + \frac{2}{(s+4)} \tag{16}$$

Following the conventional method

$$F_2(s) = \frac{(4s+10)}{(s-2)(s+1)(s+4)} = \frac{A}{(s-2)} + \frac{B}{(s+1)} + \frac{C}{(s+4)}$$

For

$$s = 2, A = \frac{8+10}{(3)(6)} = 1$$

For s = -1, B = -2/3

For

$$s = -4, C = \frac{-16+10}{(-6)(-3)} = \left(-\frac{1}{3}\right)$$

Thus

$$F_2(s) = \frac{1}{(s-2)} - \frac{2}{3(s+1)} - \frac{1}{3(s+4)} \tag{17}$$

Combining (16) ands (17), we get

$$T(s) = \frac{2}{(s-3)} + \frac{3}{(s-2)} + \frac{4}{3(s+1)} + \frac{2}{(s+3)} + \frac{5}{3(s+4)} \tag{18}$$

This is the same as (14), but NDD method has shortened the procedure and made calculations easier.

In Example 4, $D_1(s)$ cannot be factored. Therefore, no partial fraction expansion exists.

VI. NDD TO NNDD TRANSFORMATION

$T_1(s)$ can be converted back to the original T(s) by cancelling some or all the common factors between $N_1(s)$ and $D_1(s)$.

VII. PROPERTY OF NDD FUNCTION

An NDD function becomes NNDD if s replaced by $F(x) \neq x$.

Proof: Since, $T_1(s)$ is NDD function,

$$N_1(s) = \frac{d}{ds} D_1(s).$$

Substituting $s = F(x)$ and $ds = F'(x)dx$

$$N_1(x) = \left[\frac{1}{F'(x)}\right] \frac{d}{dx} D_1(x)$$

$$\neq \frac{d}{dx} D_1(x).$$

except when $F(x) = x$. Thus, the new function is an NNDD function.

VIII. Conclusion

It has been shown that a class of NNDD functions (a ratio of finite length polynomials) can be converted into an NDD function by multiplying both the numerator and denominator by a suitable function. Expression for this function has been has been mentioned. Finally a property of NDD function has been stated and proved.

Reference

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