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Iterated Function System of Generalized Non – Reducible Farey N – Subsequence of order (4, 6...) by using HB Operator

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Abstract:-
This paper establishes Non - Reducible Farey N - Subsequence of order (4, 6,...). This research formulates iterated function systems by using HB Operator and additionally shows the generalized Non - Reducible Farey N - Subsequence of order (4, 6,...).
Keywords:-
Arithmetic Mediant, Farey 'N' subsequence, Hausdorff dimension, Invariant Measure, Iterated Function System, Markov Operator, Non - Reducible Farey Sequence.

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I. Introduction

The Farey sequence is an example that has its inception for all intents and common numbers. The Farey sequence was so named after the British born geologist, John Farey (1766-1826). Given a sequence F_N where b, d and $b + d$ are all less than N, what Farey noticed is that if two fractions $\frac{a}{b}$ and $\frac{c}{d}$ were combined in the way $\frac{a+c}{b+d}$, the resulting fraction was also in the series. Farey was not able to prove this but prolific French mathematician Augustin Cauchy (1789-1857) had the option to give a proof in 1816 and published in Exerices de mathematiques [1,3,5,6,8,9].

This paper is organized as follows: Section 2 Basic definitions with example. Section 3 iterated function system of Non – Reducible Farey N – subsequence of order 4. Section 4 Research formulates iterated function system of Non – Reducible Farey N – subsequence of order 6. Section 5 the researcher shows the iterated function system of Non – Reducible Farey N – subsequence of order (2m-2).

II. Preliminaries

Throughout this paper we study the Non – Reducible Farey N – subsequence of order (4, 6,...)

Definition 2.1: Farey sequence [2]
A Farey sequence F_n is the set of rational numbers $\frac{p}{q}$ with p and q coprime, with $0 < p < q < n$, ordered by size.

Example 2.1.1:

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}\, \frac{1}{1}\right\}$$
$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

Definition 2.2: Farey N – subsequence [2]
The subsequence of farey sequence of order N whose denominators is equal to N is named as Farey N – subsequence and denoted by $\langle F_N' \rangle$.

$$\langle F_N' \rangle = \left\{\frac{u_i}{N} / \, 0 \le u_i \le N, 0 \le i \le N\right\}$$

Definition 2.3: Non - Reducible Farey Sequence [2]
The Sequence of non-reduced fractions with denominators not exceeding N listed in order of their size is called Non - Reducible Farey Sequence of order N. It is denoted by $\widetilde{F_N}$.

Example 2.3.1:
The Non - Reducible Farey Sequence of order 4 is

$$\tilde{F}_4 = \left\{\frac{0}{1} = \frac{0}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} = \frac{2}{4}, \frac{2}{3}, \frac{3}{4}, \frac{4}{4} = \frac{1}{1}\right\}$$

Definition 2.4: Non - Reducible Farey N – Subsequence [2]
The Sequence of non-reduced Farey fractions with denominators equal to the order of the size N is called Non Reducible Farey N – Subsequence. It is denoted by $\tilde{F}_N$.

Example 2.4.1:
The Non - Reducible Farey N - Subsequence of order 4 is

$$\tilde{F}_4 = \left\{\frac{0}{4}, \frac{1}{4},, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = \frac{1}{1}\right\}$$

Definition 2.5: Arithmetic Mediant Property [2]
The Arithmetic Mediant is the fraction $\frac{a+c}{b+d}$ computed from the two fractions $\frac{a}{b}$ and $\frac{c}{d}$. If the two fractions are such that $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Definition 2.6: Iterated Function System [4]
An Iterated Function System is a limited set of contraction mappings on a complete metric space. Symbolically, $\{f_i : X \to X,\ where\ i = 1,2, \ldots..N\}$, $N \epsilon \mathbb{N}$ is an iterated function system if each f_i is a contraction on the complete metric space X.

Definition 2.7: HB Operator [7]
The Hutchison – Barnsley operator (HB operator) of the IFS is a function $G : X \to X$ defined by
$G(B) = \cup_{n=1}^{N} g_n(B)$, for all $B \in X$

III. Iterated Function System of Non – Reducible Farey N – Subsequence of order 4

The Non – Reducible Farey N – Subsequence of order 4 may be obtained by the following iterated function system using HB operator.

$$g_1(x) = \frac{x}{4} \quad ; \quad g_3(x) = \frac{x+2}{4}$$
$$g_2(x) = \frac{x+1}{4} \quad ; \quad g_4(x) = \frac{x+3}{4}$$

The contraction factor is $\frac{1}{4}$. The HB operator is

$$G(B) = \cup_{k=1}^{4} g_k(B)$$

Then the IFS is

$$B_{n+1} = G(B_n)\ ; \quad B_0 \in \left\{\frac{0}{1}, \frac{1}{1}\right\}\ ; \ n = 0,1,2,\cdots\cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \to (3.1)$$

Non – Reducible Farey N – Subsequence of order 4 from equation 3.1 we obtain obviously,
Equation 3.1$\Rightarrow$ $\quad B_{n+1} = G(B_n)$
put n = 0 , we get

$$B_1 = G(B_0)\ ; \quad B_0 \in \left\{\frac{0}{1}, \frac{1}{1}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \to (3.2)$$

$\Rightarrow G(B_0) = g_1(B_0) \cup g_2(B_0) \cup g_3(B_0) \cup g_4(B_0)$
Consider,

$$\left.\begin{matrix} g_1(B_0) = \left\{\frac{0}{4}, \frac{1}{4}\right\} \ ; \quad g_3(B_0) = \left\{\frac{2}{4}, \frac{3}{4}\right\} \\ g_2(B_0) = \left\{\frac{1}{4}, \frac{2}{4}\right\} \ ; \quad g_4(B_0) = \left\{\frac{3}{4}, \frac{4}{4} = 1\right\} \end{matrix}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \to (3.3)$$

Equation 3.3 substitute in equation 3.2

$$B_1 = \left\{\frac{0}{4}, \frac{1}{4}\right\} \cup \left\{\frac{1}{4}, \frac{2}{4}\right\} \cup \left\{\frac{2}{4}, \frac{3}{4}\right\} \cup \left\{\frac{3}{4}, \frac{4}{4} = 1\right\}$$
$$B_1 = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = 1\right\}$$

$\therefore$ The sequences B_1 are $\left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = 1\right\}$

Put n = 1, we get

$$B_2 = G(B_1) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \to (3.4)$$

$\Rightarrow G(B_1) = g_1(B_1) \cup g_2(B_1) \cup g_3(B_1) \cup g_4(B_1)$
Consider,

$$\left.\begin{aligned} g_1(B_1) &= \left\{\frac{0}{16},\frac{1}{16},\frac{2}{16},\frac{3}{16},\frac{4}{16}\right\} \\ g_2(B_1) &= \left\{\frac{1}{16},\frac{5}{16},\frac{6}{16},\frac{7}{16},\frac{8}{16}\right\} \\ g_3(B_1) &= \left\{\frac{2}{16},\frac{9}{16},\frac{10}{16},\frac{11}{16},\frac{12}{16}\right\} \\ g_4(B_1) &= \left\{\frac{3}{16},\frac{13}{16},\frac{14}{16},\frac{15}{16},\frac{16}{16}=1\right\} \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \to (3.5)$$

Equation 3.5 substitute in equation 3.4

$$B_2 = \left\{\frac{0}{16},\frac{1}{16},\frac{2}{16},\frac{3}{16},\frac{4}{16}\right\} \cup \left\{\frac{1}{16},\frac{5}{16},\frac{6}{16},\frac{7}{16},\frac{8}{16}\right\} \cup \left\{\frac{2}{16},\frac{9}{16},\frac{10}{16},\frac{11}{16},\frac{12}{16}\right\} \cup \left\{\frac{3}{16},\frac{13}{16},\frac{14}{16},\frac{15}{16},\frac{16}{16}=1\right\}$$

$$B_2 = \left\{\frac{0}{16},\frac{1}{16},\frac{2}{16},\frac{3}{16},\frac{4}{16},\frac{5}{16},\frac{6}{16},\frac{7}{16},\frac{8}{16},\frac{9}{16},\frac{10}{16},\frac{11}{16},\frac{12}{16},\frac{13}{16},\frac{14}{16},\frac{15}{16},\frac{16}{16}=1\right\}$$

$\therefore$ The sequences B_2 are $\left\{\frac{0}{16},\frac{1}{16},\frac{2}{16},\frac{3}{16},\frac{4}{16},\frac{5}{16},\frac{6}{16},\frac{7}{16},\frac{8}{16},\frac{9}{16},\frac{10}{16},\frac{11}{16},\frac{12}{16},\frac{13}{16},\frac{14}{16},\frac{15}{16},\frac{16}{16}=1\right\}$

Hence

$$\therefore B_1 \Rightarrow G(B_0) = \left\{\frac{0}{4},\frac{1}{4},\frac{2}{4},\frac{3}{4},\frac{4}{4}=1\right\}$$

$$B_2 \Rightarrow G(B_1) = \left\{\frac{0}{16},\frac{1}{16},\frac{2}{16},\frac{3}{16},\frac{4}{16},\frac{5}{16},\frac{6}{16},\frac{7}{16},\frac{8}{16},\frac{9}{16},\frac{10}{16},\frac{11}{16},\frac{12}{16},\frac{13}{16},\frac{14}{16},\frac{15}{16},\frac{16}{16}=1\right\}$$

$$\vdots$$

$$B_{n+1} \Rightarrow G(B_n) = \left\{\frac{0}{4^n},\frac{1}{4^n},\frac{2}{4^n},\frac{3}{4^n},\cdots\cdots\frac{4^n-4}{4^n},\frac{4^n-3}{4^n},\frac{4^n-2}{4^n},\frac{4^n-1}{4^n},\frac{4^n}{4^n}=1\right\}$$

Thus the Non – Reducible Farey N – Subsequence of order 4 is

$$\tilde{F}_{4^n} = B_n \quad ; \quad n = 1,2,3,\cdots\cdots\infty$$

IV. Iterated Function System of Non – Reducible Farey N – Subsequence of order 6

The Non – Reducible Farey N – Subsequence of order 6 may be obtained by the following iterated function system using HB operator.

$$g_1(x) = \frac{x}{6} \quad ; \quad g_4(x) = \frac{x+3}{6}$$
$$g_2(x) = \frac{x+1}{6} \quad ; \quad g_5(x) = \frac{x+4}{6}$$
$$g_3(x) = \frac{x+2}{6} \quad ; \quad g_6(x) = \frac{x+5}{6}$$

The contraction factor is $\frac{1}{6}$. The HB operator is

$$G(B) = \cup_{k=1}^{6} g_k(B)$$

Then the IFS is

$$B_{n+1} = G(B_n)\,; \quad B_0 \in \left\{\frac{0}{1},\frac{1}{1}\right\} \;;\; n = 0,1,2,\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots \to (4.1)$$

Non – Reducible Farey N – Subsequence of order 6 from equation 4.1 we obtain obviously,

Equation 4.1 $\Rightarrow$ $B_{n+1} = G(B_n)$

Put n = 0, we get

$$B_1 = G(B_0)\,; \quad B_0 \in \left\{\frac{0}{1},\frac{1}{1}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \to (4.2)$$

$$\Rightarrow G(B_0) = g_1(B_0) \cup g_2(B_0) \cup g_3(B_0) \cup g_4(B_0) \cup g_5(B_0) \cup g_6(B_0)$$

Consider,

$$\left.\begin{aligned} g_1(B_0) &= \left\{\frac{0}{6},\frac{1}{6}\right\} \;;\quad g_4(B_0) = \left\{\frac{3}{6},\frac{4}{6}\right\} \\ g_2(B_0) &= \left\{\frac{1}{6},\frac{2}{6}\right\} \;;\quad g_5(B_0) = \left\{\frac{4}{6},\frac{5}{6}\right\} \\ g_3(B_0) &= \left\{\frac{2}{6},\frac{3}{6}\right\} \;;\quad g_6(B_0) = \left\{\frac{5}{6},\frac{6}{6}=1\right\} \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots \to (4.3)$$

Equation 4.3 substitute in equation 4.2

$$B_1 = \left\{\frac{0}{6},\frac{1}{6}\right\} \cup \left\{\frac{1}{6},\frac{2}{6}\right\} \cup \left\{\frac{2}{6},\frac{3}{6}\right\} \cup \left\{\frac{3}{6},\frac{4}{6}\right\} \cup \left\{\frac{5}{6},\frac{6}{6}=1\right\}$$

$$B_1 = \left\{\frac{0}{6},\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6},\frac{6}{6}=1\right\}$$

$\therefore$ The sequences B_1 are $\left\{\frac{0}{6},\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6},\frac{6}{6}=1\right\}$

Put n = 1, we get

$$B_2 = G(B_1) \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \to (4.4)$$

$$\Rightarrow G(B_1) = g_1(B_1) \cup g_2(B_1) \cup g_3(B_1) \cup g_4(B_1) \cup g_5(B_1) \cup g_6(B_1)$$

Consider,

$$\left.\begin{aligned} g_1(B_1) &= \left\{\frac{0}{36},\frac{1}{36},\frac{2}{36},\frac{3}{36},\frac{4}{36},\frac{5}{36},\frac{6}{36}\right\} \\ g_2(B_1) &= \left\{\frac{1}{36},\frac{7}{36},\frac{8}{36},\frac{9}{36},\frac{10}{36},\frac{11}{36},\frac{12}{36}\right\} \\ g_3(B_1) &= \left\{\frac{2}{36},\frac{13}{36},\frac{14}{36},\frac{15}{36},\frac{16}{36},\frac{17}{36},\frac{18}{36}\right\} \\ g_4(B_1) &= \left\{\frac{3}{36},\frac{19}{36},\frac{20}{36},\frac{21}{36},\frac{22}{36},\frac{23}{36},\frac{24}{36}\right\} \\ g_5(B_1) &= \left\{\frac{4}{36},\frac{25}{36},\frac{26}{36},\frac{27}{36},\frac{28}{36},\frac{29}{36},\frac{30}{36}\right\} \\ g_6(B_1) &= \left\{\frac{5}{36},\frac{31}{36},\frac{32}{36},\frac{33}{36},\frac{34}{36},\frac{35}{36},\frac{36}{36}=1\right\} \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \to (4.5)$$

Equations 4.5 substitute in equation 4. 4

$$B_2 = \left\{\frac{0}{36},\frac{1}{36},\frac{2}{36},\frac{3}{36},\frac{4}{36},\frac{5}{36},\frac{6}{36}\right\} \cup \left\{\frac{1}{36},\frac{7}{36},\frac{8}{36},\frac{9}{36},\frac{10}{36},\frac{11}{36},\frac{12}{36}\right\} \cup \left\{\frac{2}{36},\frac{13}{36},\frac{14}{36},\frac{15}{36},\frac{16}{36},\frac{17}{36},\frac{18}{36}\right\}$$
$$\cup \left\{\frac{3}{36},\frac{19}{36},\frac{20}{36},\frac{21}{36},\frac{22}{36},\frac{23}{36},\frac{24}{36}\right\} \cup \left\{\frac{4}{36},\frac{25}{36},\frac{26}{36},\frac{27}{36},\frac{28}{36},\frac{29}{36},\frac{30}{36}\right\} \cup \left\{\frac{5}{36},\frac{31}{36},\frac{32}{36},\frac{33}{36},\frac{34}{36},\frac{35}{36},\frac{36}{36}=1\right\}$$

$$B_2 = \left\{\frac{0}{36},\frac{1}{36},\frac{2}{36},\frac{3}{36},\frac{4}{36},\cdots\cdots\cdots\frac{30}{36},\frac{31}{36},\frac{32}{36},\frac{33}{36},\frac{34}{36},\frac{35}{36},\frac{36}{36}=1\right\}$$

$\therefore$ The sequences B_2 are $\left\{\frac{0}{36},\frac{1}{36},\frac{2}{36},\frac{3}{36},\frac{4}{36},\cdots\cdots\cdots\frac{30}{36},\frac{31}{36},\frac{32}{36},\frac{33}{36},\frac{34}{36},\frac{35}{36},\frac{36}{36}=1\right\}$

Hence

$$\therefore B_1 \Rightarrow G(B_0) = \left\{\frac{0}{6},\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6},\frac{6}{6}=1\right\}$$
$$B_2 \Rightarrow G(B_1) = \left\{\frac{0}{36},\frac{1}{36},\frac{2}{36},\frac{3}{36},\frac{4}{36},\cdots\cdots\cdots\frac{30}{36},\frac{31}{36},\frac{32}{36},\frac{33}{36},\frac{34}{36},\frac{35}{36},\frac{36}{36}=1\right\}$$
$$\vdots$$
$$B_{n+1} \Rightarrow G(B_n) = \left\{\frac{0}{6^n},\frac{1}{6^n},\frac{2}{6^n},\frac{3}{6^n},\cdots\cdots\frac{6^n-6}{6},\frac{6^n-5}{6},\frac{6^n-4}{6},\frac{6^n-3}{6},\frac{6^n-2}{6},\frac{6^n-1}{6},\frac{6^n}{6^n}=1\right\}$$

Thus the Non – Reducible Farey N – Subsequence of order 6 is

$$\tilde{F}_{6^n} = B_n \quad ; \quad n = 1, 2, 3, \cdots\cdots\infty$$

V. Iterated Function System of Non – Reducible Farey N – Subsequence of order (2m-2)

The Non – Reducible Farey N – Subsequence of order (2m-2); $3 \le m < \infty$ may be obtained by the following iterated function system using HB operator.

$$g_1(x) = \frac{x}{2m-2}$$
$$g_2(x) = \frac{x}{2m-2} + \frac{1}{2m-2}$$
$$g_3(x) = \frac{x}{2m-2} + \frac{2}{2m-2}$$
$$\vdots$$
$$g_N(x) = \frac{x}{2m-2} + \frac{(N-1)}{2m-2} \; ; \qquad N = 1,2,3,\cdots\cdots\infty$$

The contraction factor is $\frac{1}{2m-2}$. The HB operator is

$$G(B) = \cup_{k=1}^{N} g_k(B)$$

Then the IFS is

$$B_{n+1} = G(B_n)\,; \quad B_0 \in \left\{\frac{0}{1},\frac{1}{1}\right\} \; ; \; n = 0,1,2,\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \to (5.1)$$

Non – Reducible Farey N – Subsequence of order (2m-2) from equation 5.1 we obtain obviously,

$$\therefore B_1 \Rightarrow G(B_0) = g_1(B_0) \cup g_2(B_0) \cup g_3(B_0) \cup \cdots\cdots \cup g_k(B_0)$$
$$B_2 \Rightarrow G(B_1) = g_1(B_1) \cup g_2(B_1) \cup g_3(B_1) \cup \cdots\cdots \cup g_k(B_1)$$
$$\vdots$$
$$B_{n+1} \Rightarrow G(B_n) = \left\{\begin{matrix} \frac{0}{(2m-2)^n},\frac{1}{(2m-2)^n},\frac{2}{(2m-2)^n},\frac{3}{(2m-2)^n},\cdots\cdots\frac{(2m-2)^n-3}{(2m-2)^n},\frac{(2m-2)^n-2}{(2m-2)^n}, \\ \frac{(2m-2)^n-1}{(2m-2)^n},\frac{(2m-2)^n}{(2m-2)^n} = 1 \end{matrix}\right\}$$

Thus the Non – Reducible Farey N – Subsequence of order $(2m-2); 3 \le m < \infty$, is

$$\tilde{F}_{(2m-2)^n} = B_n \quad ; \quad n = 1, 2, 3, \cdots\cdots\infty$$

VI. Conclusion

We have established Non - Reducible Farey N - Subsequence of order (4, 6,...). We have been formulated iterated function systems by using HB Operator and also additionally shown their generalized Non - Reducible N - Subsequence of order (4, 6,...).

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