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Angle Trisector in the Triangle

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Abstract:
Background: *In general, a lot of discussion on the line for discussing the angle bisector, in this paper will discuss about the angle trisector (a line that divides an angle into three equal parts). The discussed is about the side lengths of the angular trisector and the ratio of the area of the corner trisector formed from each corner of the triangle. The proof is done using a very simple method, namely by using the concept of the height line on the triangle and the trigonometric ratio of the triangle.*
Keywords: *Angle trisector, The side length of the trisector,Comparison of the area of the trisector triangle.*

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I. Introduction

The discussion of thetrisector previously tends to be a theorem stated by a Professor of Mathematics at Haverford Collage named Frank Morley (1899), the theorem is theorem Morley[10]. According Wall (2008), Morley stated that if there is any triangle formed by the trisector at each corner, then there are three intersections of the two adjacent tricors to form Morley's triangle[14].Several proofs of Morley's theorem with different points of view have been found by several mathematicians such asDonaloto, (2013)[4] and Strongebridge, (2009)[13]. Furthermore, the development of Morley's theorem is also discussed by Barutu et al. which discusses the development of Morley's theorem on rectangles[2]. Then Kuruklis, (2014) provides a case related to the Morley theorem, namely the Morley triangle can be formed from any external angular trisector of any, triangle[6]. Then, the development of Morley's theorem on the external angle trisector of a triangle, was developed by Husnawhich discusses the theorem of the Morley Outer Trisector in triangles and rectangle[5].

In this paper, the researcher is interested in discussing the length of the angular trisector in a triangle and also discussing the comparison of the area of the trisector triangle to the triangle. The Proof it will be done by using a simple concept, namely the concept of the height line in the triangle and the concept of trigonometry, with usethe sides and angles of triangles are knownon triangle, which is discussed in Mashadi, (2015)[7] and Mashadi, (2016)[8].So that it can be determined the length of the angular trisector in a triangle and also discussing the comparison of the area of the trisector triangle to the triangleif the sides and angles of triangles are known**.**

II. The Triangle's Trisector

In addition to dividing the angle into two equals, in a triangle, if each vertex is drawn two lines to the side in front of it, it can divide the angle into three equal.
Definition 2.1. (Angle Trisector) has two dividing lines that divide the angle into three equal parts.

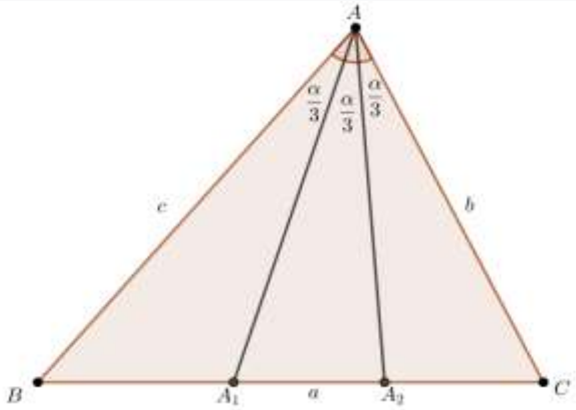

Figure1.$\Delta ABC$with AA_1andAA_2 is trisector ofangle A.

On figure 1, AA_1 and AA_2 is angle trisector line at angle *A*, that divides an angle into three equal parts. Trisectors are often discussed in a theorem, which is a theorem called the Morley theorem. Morley's theorem is one of the most interesting aspects of geometry in the twentieth century[3]. Look at Figure 2, if there are at each corner in the form of a trisector that divides each inner corner into three equal parts.
Suppose the $\angle A$ given the names a_1 and a_2, the trisector $\angle B$ was given the name b_1 and b_2, and the trisector $\angle C$ was given the name c_1 and c_2. Let the points *D*, *E* and *F* be the intersection points between the lines a_1 and b_1, a_2 and c_2, b_2 and c_1. If the three points of intersection are connected, an equilateral triangle *DEF* is formed(figure 2), as shown in the illustrationthe following Theorem.

Theorem2.1. (Morley's Theorem) For instance ΔABC , the adjacent inner trisector will intersect at one point, namely points *D*, *E* and *F*, if the intersecting points are connected then an equilateral triangle is namely ΔDEF.

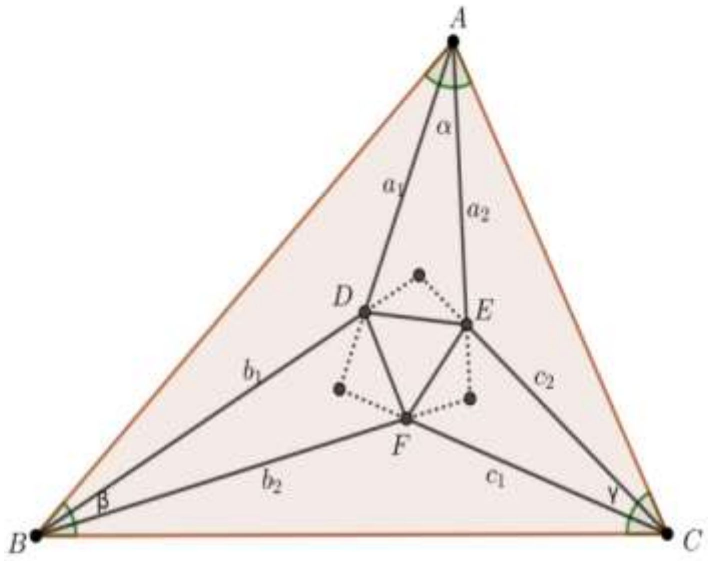

Figure 2.ΔDEF is Morley' triangle of ΔABC

The length of Morley's triangle has been discussed in several writings which are found in different ways, including in, writing[4], The length of Morley's triangle with, $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$ formulated $8R sin(\alpha) \sin(\gamma) \sin(\beta)$.

III. Trisector Side Length Of Triangle

On the ΔABC, if the corner is formed an angular trisector, there are two sides of the trisector which divide the angle into three equal parts. Trisector AA_1 and AA_2 divide the angle A into three equal parts, Trisector BB_1 and BB_2 divide the angle B into three equal parts and Trisector CC_1 and CC_2 divide the angle C into three equal parts. The Theorem as following.

Theorem 3.1. On the ΔABC, formed a trisector at the angle of the triangle, AA_1 and AA_2 divide the angle A with $BC = a, AC = b, AB = c$ and $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$, than length of AA_1 and AA_2 is

$$AA_1 = \frac{2L}{\alpha \sin\left(\frac{\alpha}{3} + \beta\right)}$$

$$AA_2 = \frac{2L}{\alpha \sin\left(\frac{\alpha}{3} + \gamma\right)}.$$

PROOF: Look at picture 3,

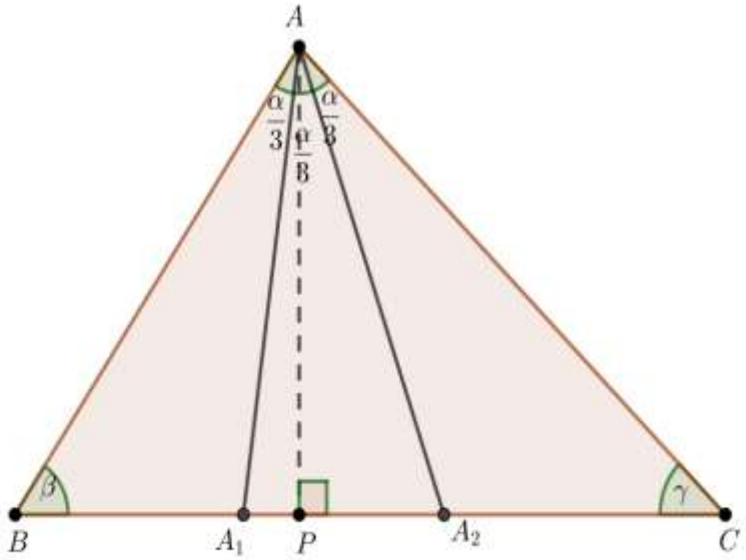

Figure 3.$\Delta ABC$with $AA_1$and AA_2 is trisector of angle A and shape the line heightAP.

Shape the line height onΔABC, from the corner$\angle A$to the side$A$, on point P . SeeΔAA_1P obtained

$$\angle A_1AP = 180^{\circ} - \left(90^{\circ} + \left(\frac{\alpha}{3} + \beta\right)\right)$$
$$= 90^{\circ} - \left(\frac{\alpha}{3} + \beta\right)$$

Furthermore, using a trigonometric ratio onΔAA_1P , when we get

$$\cos \angle A_1AP = \frac{AP}{AA_1}$$

Substitution value$\angle A_1AP$and the height of the triangle is,obtained

$$\cos\left(90^{\circ} - \left(\frac{\alpha}{3} + \beta\right)\right) = \frac{\frac{2}{a}\sqrt{s(s-a).(s-b).(s-c)}}{AA_1}$$

$$AA_1 = \frac{2\sqrt{s(s-a).(s-b).(s-c)}}{a.\cos\left(90^{\circ} - \left(\frac{\alpha}{3} + \beta\right)\right)}$$

By using a formula$\cos(90° - \alpha) = \sin \alpha$and the formula for the area of a triangle, if all three sides are known, it is obtained

$$AA_1 = \frac{2L}{a\sin\left(\frac{\alpha}{3} + \beta\right)}$$

Next will be determined the side length AA_2, look ΔAA_2C obtained

$$\angle AA_2C = 180^{\circ} - \left(\frac{\alpha}{3} + \gamma\right)$$

Because ΔAA_2B straightener ΔAA_2C, is obtained

$$\angle AA_2B = 180^{\circ} - \left(180^{\circ} - \left(\frac{\alpha}{3} + \gamma\right)\right)$$

$$= \frac{\alpha}{3} + \gamma$$

to high reuse ΔABC, see ΔAA_2P it is obtained $\angle A_2AP$,

$$\angle A_2AP = 90^{\circ} - \left(\frac{\alpha}{3} + \gamma\right)$$

Next by using the cosine rule on ΔAA_2P, then

$$\cos \angle A_2AP = \frac{AP}{AA_2}$$

Substitutionvalue $\angle A_2AP$, and the height of the triangle is AP. obtained

$$\cos\left(90^{\circ} - \left(\frac{\alpha}{3} + \gamma\right)\right) = \frac{\frac{2}{a}\sqrt{s(s-a).(s-b).(s-c)}}{AA_2}$$

$$AA_2 = \frac{2\sqrt{s(s-a).(s-b).(s-c)}}{a.\cos\left(90^{\circ} - \left(\frac{\alpha}{3} + \gamma\right)\right)}$$

By using a formula $\cos(90^{\circ} - \alpha) = \sin\alpha$ and the formula for the area of a triangle, if all three sides are known, it is obtained

$$AA_2 = \frac{2L}{a\sin\left(\frac{\alpha}{3} + \gamma\right)}$$

By using the same method, for determine the length of the trisector and we get the trident at the other corner of the triangle as in Figure 4.

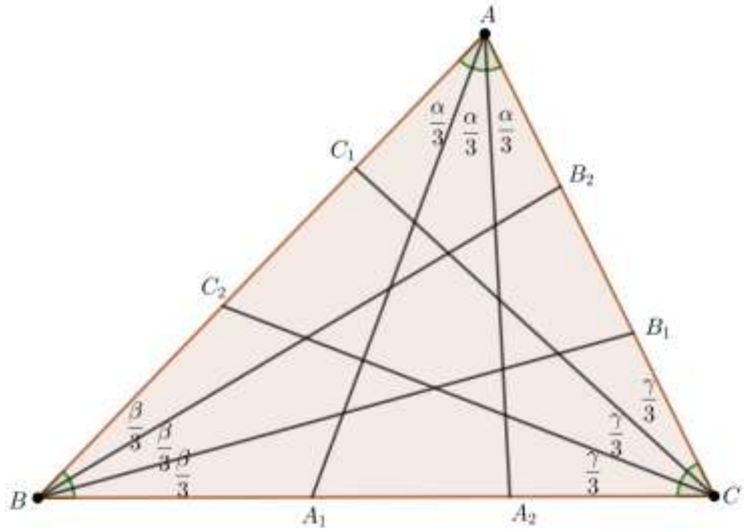

Figure 4.ΔABC with a trisector at each corner of the triangle.

$$BB_1 = \frac{2L}{b\sin\left(\frac{\beta}{3}+\gamma\right)}$$

$$BB_2 = \frac{2L}{b\sin\left(\frac{\beta}{3}+\alpha\right)}$$

$$CC_1 = \frac{2L}{c\sin\left(\frac{\gamma}{3}+\alpha\right)}$$

$$CC_2 = \frac{2L}{c\sin\left(\frac{\gamma}{3}+\beta\right)}$$

IV. Comparison The Area Of A Trisector Triangle

On the ΔABC, if the corner is formed an angular trisector,then a trident triangle will be formed at each of these angles, namely angle A is ΔABA_1,ΔAA_1A_2 and ΔAA_2C, angle B is ΔBCB_1,ΔBB_1B_2 and ΔBB_2A and angle C is ΔCAC_1,ΔCC_1C_2 and ΔCC_2B. In the following, we will show the comparison of the area of the trisector triangle from the angle A, as in the following theorem.

Theorem 4.1. On the ΔABC if the corner is formed an angular trisector A then the ratio of the area of the trisector triangle is

$$L\Delta ABA_1 : L\Delta AA_1A_2 : L\Delta AA_2C = \sin\left(\frac{\alpha}{3}+\gamma\right)\sin\gamma : \sin\beta\sin\gamma : \sin\left(\frac{\alpha}{3}+\beta\right)\sin\beta$$

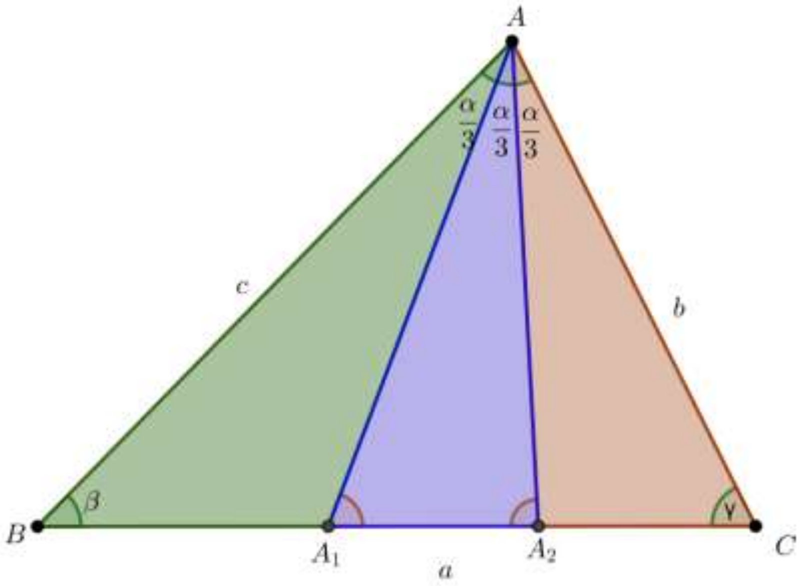

Figure 5..$\Delta ABC$with the ratio of the area of the trisector triangle of $\angle A$

Proof. View the trisector triangle from$\angle A$is ΔABA_1,$\Delta AA_1A_2$and ΔAA_2C,the three triangles are the same height because they are atΔABCthen the ratio of the area of the trisector triangle at $\angle A$to the respective bases is obtained.

$$\frac{L\Delta ABA_1}{L\Delta AA_1A_2} = \frac{\frac{1}{2}\cdot t\cdot BA_1}{\frac{1}{2}\cdot t\cdot A_1A_2} = \frac{BA_1}{A_1A_2}$$

$$\frac{L\Delta AA_1A_2}{L\Delta AA_2C} = \frac{\frac{1}{2}\cdot t\cdot A_1A_2}{\frac{1}{2}\cdot t\cdot A_2C} = \frac{A_1A_2}{A_2C}$$

So that the ratio of the area of the trisector triangle will be shown by showing the ratio of each side, namely

$$BA_1 : A_1A_2 : A_2C .$$

The first step will be shown a side comparison$BA_1 : A_1A_2$, seeΔABA_1, by using the sine rule on

$$\frac{BA_1}{\sin\angle BAA_1} = \frac{AA_1}{\sin\angle ABA_1}$$

$$\sin\angle BAA_1 = BA_1\cdot\frac{\sin\angle ABA_1}{AA_1}$$

From the ΔAA_1A_2, also obtained.

$$\frac{A_1A_2}{\sin\angle A_1AA_2} = \frac{AA_1}{\sin\angle AA_2A_1}$$

$$\sin\angle A_1AA_2 = A_1A_2\cdot\frac{\sin\angle AA_2A_1}{AA_1}$$

Because the$\angle BAA_1 = \angle A_1AA_2$is obtained

$$\frac{\sin\angle BAA_1}{\sin\angle A_1AA_2}=\frac{BA_1\sin\angle ABA_1}{A_1A_2\sin\angle AA_2A_1}$$

$$\frac{BA_1}{A_1A_2}=\frac{\sin AA_2A_1}{\sin\angle ABA_1}$$

$$\frac{BA_1}{A_1A_2}=\frac{\sin\left(\frac{\alpha}{3}+\gamma\right)}{\sin\beta}$$

In the same way the side ratio is determined $A_1A_2 : A_2C$ by using the sinus rule on ΔAA_1A_2 and ΔAA_2C so obtained.

$$\frac{A_1A_2}{A_2C}=\frac{\sin\gamma}{\sin\left(\frac{\alpha}{3}+\beta\right)}$$

Then using cross multiplication is obtained $BA_1 : A_1A_2 : A_2C$, is

$$\frac{BA_1}{A_1A_2}=\frac{\sin\left(\frac{\alpha}{3}+\gamma\right)}{\sin\beta}\cdot\frac{\sin\gamma}{\sin\gamma}$$

$$\frac{A_1A_2}{A_2C}=\frac{\sin\gamma}{\sin\left(\frac{\alpha}{3}+\beta\right)}\cdot\frac{\sin\beta}{\sin\beta}$$

$$BA_1 : A_1A_2 : A_2C=\sin\left(\frac{\alpha}{3}+\gamma\right)\sin\gamma : \sin\beta\sin\gamma : \sin\left(\frac{\alpha}{3}+\beta\right)\sin\beta$$

Thus, the ratio of the area of the trisector triangle from the angle is obtained

$$L\Delta ABA_1 : L\Delta AA_1A_2 : L\Delta AA_2C=\sin\left(\frac{\alpha}{3}+\gamma\right)\sin\gamma : \sin\beta\sin\gamma : \sin\left(\frac{\alpha}{3}+\beta\right)\sin\beta$$

Using the same way determine the area ratio of the triangle trisector at angle A , we get the rasio of the area of the triangle trisector angle B dan C is

$$L\Delta BCB_1 : L\Delta BB_1B_2 : L\Delta BB_2A=\sin\left(\frac{\beta}{3}+\alpha\right)\sin\alpha : \sin\gamma\sin\alpha : \sin\left(\frac{\beta}{3}+\gamma\right)\sin\gamma$$

$$L\Delta CAC_1 : L\Delta CC_1C_2 : L\Delta CC_2B=\sin\left(\frac{\gamma}{3}+\beta\right)\sin\beta : \sin\alpha\sin\beta : \sin\left(\frac{\gamma}{3}+\alpha\right)\sin\alpha$$

V. Conclusion

From the paper it can be concluded that the side lengths of the trisector in triangles can be calculated if the sides and angles of triangles are known, the proof is done by using a very simple concept in geometry. If the angletrisector of the triangle is known, the ratio of the area of the trisector triangle can be determined using the formula for the area of the triangle.

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