IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 6 Ser. II (Nov. – Dec. 2020), PP 10-17
www.iosrjournals.org

On The Degree of Approximation by Modified Beta Bernstein Operators.

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ABSTRACT: *In this article,we define the modified form of Bernstein type operators based on Beta function developed by Dhawal.J. Bhatt et al. recently.We have proved these operators uniform convergence on the basis of Korovkin's theorem and rate of convergence through modulus of continuity and asymptotic behaviour is shown as Vorovnoskaja type theorem.*

Date of Submission: 10-11-2020 Date of Acceptance: 25-11-2020

I. Preliminaries

Most celebrated Weierstrass approximation theorem is the milestone in the development of Approximation theory which is currently understood as the given form :

If $f:[a,b]\to \mathbb{R}$ continuous and for any $s>0$, $\exists$ an algebraic polynomial p such that

$$|f(x) - p(x)| \le s, \qquad \forall x \in [a,b]$$

Bernstein [1] gave a very sound proof of this theorem and introduced Bern- stein polynomial of degree n of the function f defined as

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \qquad (1)$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, and $0 \le k \le n$.

Lots of generalisation and improvements [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] of (1) are studied by many researchers.

Recently Dhawal J. Bhatt et al. [12] defined the following Beta-Bernstein operator

B_n :
For $f \in C[0, 1]$, $B_n : C[0, 1] \to C[0, 1]$

$$\mathfrak{B}_{\mathfrak{n}}(f;x) = \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k}{n}\right)$$

where $P_{n,k}(x) = (n/k)\ \beta(nx\beta(nx+1,2n-k-nx+1)$, $x \in [0, 1]$, $\beta(a, b)$ is the Beta

function defined as

$P_{n,k}(x) = \binom{n}{k} \frac{\beta(nx+k+1,2n-k-nx+1)}{\beta(nx+1,n-nx+1)}$, $x \in [0,1]$, $\beta(a,b)$ is the Beta

function defined as $\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt \quad (a,b>0)$

Now, we define an operator as substituting $\frac{k}{n}$ by $\frac{k}{n+1}$ in (2) so this newly

defined operator

$$\mathfrak{B}^{*}_{\mathfrak{n}}(f;x) = \sum_{k=o}^{n} P_{n,k}(x) f\left(\frac{k}{n+1}\right)$$

where $P_{n,k}(x) = \binom{n}{k} \frac{\beta(nx+k+1,2n-k-nx+1)}{\beta(nx+1,n-nx+1)}$, $x \in [0,1]$, $\beta(a,b)$ is the Beta function defined as $\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt \quad (a,b>0)$

It is obvious that the introduced operator (3) is a positive linear operator.

Auxiliary Results

Lemma 2.1. *[12] If a,b > 0 and n and k are nonnegative integers, we have the following results :*

(i) *For* $0 \le k \le n$, $\sum_{k=0}^{n} \binom{n}{k} \frac{\beta(a+k,b+n-k)}{\beta(a,b)} = 1$

(ii) *For* $1 \le k \le n$, $\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(a+k,b+n-k)}{\beta(a,b)} = \frac{a}{a+b}$

(iii) *For* $1 \le k \le n$, $\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(a+k,b+n-k)}{\beta(a,b)} \cdot \frac{k}{n} = \frac{(n-1)(a+1)a}{n(a+b+1)(a+b)} + \frac{a}{n(a+b)}$

(iv) *For* $1 \le k \le n$, $\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(a+k,b+n-k)}{\beta(a,b)} \cdot \frac{k^2}{n^2} = \frac{(n-1)(n-2)(a+2)(a+1)a}{n^2(a+b+2)(a+b+1)(a+b)} + \frac{3(n-1)(a+1)a}{n^2(a+b+1)(a+b)} + \frac{a}{n^2(a+b)}$

(v) *For* $1 \le k \le n$, $\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(a+k,b+n-k)}{\beta(a,b)} \cdot \frac{k^3}{n^3} = \frac{(n-1)(n-2)(n-3)(a+3)(a+2)(a+1)a}{n^3(a+b+3)(a+b+2)(a+b+1)(a+b)} + \frac{6(n-1)(n-2)(a+2)(a+1)a}{n^3(a+b+2)(a+b+1)(a+b)} + \frac{7(n-1)(a+1)a}{n^3(a+b+1)(a+b)} + \frac{a}{n^3(a+b)}$

Now from the above results, we compute the first four raw moments of the operator B_n^* in the following lemma.

Lemma 2.2. *If a, b > 0 and n and k are nonnegative integers, we have the following results :*

(i) $B_n^*(1;x) = 1$

(ii) $B_n^*(t;x) = \frac{n(nx+1)}{(n+1)(n+2)}$

(iii) $B_n^*(t^2;x) = \frac{n(n-1)(nx+1)(nx+2)}{(n+1)^2(n+2)(n+3)} + \frac{n(nx+1)}{(n+1)^2(n+2)}$

(iv) $B_n^*(t^3;x) = \frac{n(n-1)(n-2)(nx+3)(nx+2)(nx+1)}{(n+1)^3(n+4)(n+3)(n+2)} + \frac{3n(n-1)(nx+2)(nx+1)}{(n+1)^3(n+3)(n+2)} + \frac{n(nx+1)}{(n+1)^3(n+2)}$

(v) $B_n^*(t^4;x) = \frac{n(n-1)(n-2)(n-3)(nx+4)(nx+3)(nx+2)(nx+1)}{(n+1)^4(n+5)(n+4)(n+3)(n+2)} + \frac{6n(n-1)(n-2)(nx+3)(nx+2)(nx+1)}{(n+1)^4(n+4)(n+3)(n+2)} + \frac{7n(n-1)(nx+2)(nx+1)}{(n+1)^4(n+3)(n+2)} + \frac{n(nx+1)}{(n+1)^4(n+2)}$

Proof : (i) $B_n^*(1;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)}$

From (i) of Lemma 2.1, we obtain

$B_n^*(1;x) = 1$

(ii) $B_n^*(t;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k}{n+1} = \frac{n}{n+1} \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)}$

From (ii) of Lemma 2.1

$= \frac{n}{n+1} \cdot \frac{nx+1}{n+2}$

$= \frac{n(nx+1)}{(n+1)(n+2)}$

(iii) $B_n^*(t^2;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^2}{(n+1)^2} = \frac{n^2}{(n+1)^2} \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k}{n}$

From (iii) of Lemma 2.1

$= \frac{n(n-1)(nx+1)(nx+2)}{(n+1)^2(n+2)(n+3)} + \frac{n(nx+1)}{(n+1)^2(n+2)}$

(iv) $B_n^*(t^3;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^3}{(n+1)^3} = \frac{n^3}{(n+1)^3} \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^2}{n^2}$

From (iv) of Lemma 2.1

$= \frac{n(n-1)(n-2)(nx+3)(nx+2)(nx+1)}{(n+1)^3(n+4)(n+3)(n+2)} + \frac{3n(n-1)(nx+2)(nx+1)}{(n+1)^3(n+3)(n+2)} + \frac{n(nx+1)}{(n+1)^3(n+2)}$

(v) $\mathbf{B}^*_n(t^4; x) = \sum_n \frac{n}{k=0} \sum_{k\,k=1} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^4}{(n+1)^4}$

$= \frac{n^4}{(n+1)^4} \sum_n \frac{n^3}{k-1} \sum \beta(nx+k+1, 2n-k-nx+1) \cdot k^3$

From (v)of Lemma 2.1

$= \frac{n(n\ 1)(n\ 2)(n\ 3)(nx+4)(nx+3)(nx+2)(nx+1)}{(n+1)^4(n+5)(n+4)(n+3)(n+2)} + \frac{6n(n\ 1)(n\ 2)(nx+3)(nx+2)(nx+1)}{(n+1)^4(n+4)(n+3)(n+2)}$

$+ \frac{7n(n+1)(nx+2)(nx+1)}{(n+1)^4(n+3)(n+2)} + \frac{n(nx+1)}{(n+1)^4(n+2)}$

Now we compute the moment estimation of our defind operator B□n about x in the following lemma.

Lemma 2.3. For $x \in [0, 1]$, p^{th} $(p = 0, 1, 2)$ moments for the defined operator B□n about x are as follows :

(i) $\mathbf{B}^*_n\left((t-x)^0; x\right) = 1$

(ii) $\mathbf{B}^*_n((t-x); x)) = \frac{x(-3n-2)+n}{(n+1)(n+2)}$

(iii) $\mathbf{B}^*_n\left((t-x)^2; x\right) = \frac{x^2(7n^3+15n^2+17n+6)+x(2n^3-4n^2-2n)+(3n^2+n)}{(n+1)^2(n+2)(n+3)}$

Proof :

(i) $\mathbf{B}^*_n\left((t-x)^0; x\right) = \mathbf{B}^*_n(1; x)$

From (i) of Lemma 2.2, we have

$\mathbf{B}^*_n\left((t-x)^0; x\right) = \mathbf{B}^*_n(1; x)$

From the linearity of B□n and from (i), (ii) of lemma 2.1, we have

$\mathbf{B}^*_n((t-x); x)) = \frac{n(nx+1)}{(n+1)(n+2)} - x$

$= \frac{n\ 3nx\ 2x}{(n+1)(n+2)} = \frac{x(\ 3n\ 2)+n}{(n+1)(n+2)}$

From the linearity of B□n and from (i), (ii), (iii) of lemma 2.1, we obtain

$\mathbf{B}^*_n\left((t-x)^2; x\right) = \frac{n(n-1)(nx+1)(nx+2)}{(n+1)^2(n+2)(n+3)} + \frac{n(nx+1)}{(n+1)^2(n+2)} - 2x\frac{n(nx+1)}{(n+1)(n+2)} + x^2$

$= \frac{x^2(7n^3+15n^2+17n+6)+x(2n^3\ 4n^2\ 2n)+(3n^2+n)}{(n+1)^2(n+2)(n+3)}$

From the above similar process,it is obvious that

$\mathbf{B}^*_n\left((t-x)^3; x\right) \geq 0$

$\mathbf{B}^*_n\left((t-x)^4; x\right) \geq 0$

II. Main Result

Now,following theorem refer the *uniform convergence* of $\mathbf{B}_n^*$ for a function $f \in C[0, 1]$ with the norm

$$\|f\| = \sup_{x\in[0,1]} |f(x)|$$

Theorem 3.1. *If* $f \in C[0, 1]$, $x \in [0, 1]$ *then*

$$\|\mathbf{B}_n^*(f; x) - f(x)\| \to 0$$

uniformly as $n \to \infty$

Proof : From lemma 2.2, we have

$$\mathbf{B}_n^*(1; x) = 1$$
$$\mathbf{B}_n^*(t; x) = \frac{n(nx+1)}{(n+1)(n+2)}$$
$$\mathbf{B}_n^*(t^2; x) = \frac{n(n-1)(nx+1)(nx+2)}{(n+1)^2(n+2)(n+3)} + \frac{n(nx+1)}{(n+1)^2(n+2)}$$

It is clear that as $n \to \infty$, $\mathbf{B}_n^*(t^m; x)$ converges uniformly to $x^m (m = 0, 1, 2)$, $x \in [0, 1]$ from Korovkin's theorem [13].

III. Rate of Convergence

The modulus of continuity for $f \in C[a, b]$ is given by

$$\omega_f(\delta) \equiv \omega(f, \delta) = \sup_{\substack{x-\delta\le t\le x \\ a\le x\le b}} |f(t) - f(x)|, \qquad \delta > 0$$

The rate of convergence of the sequence of operators B _ n is estimated in the following theorem in terms of modulus of continuity of first order.

4 **Theorem 4.1.** ***If*** $f \in C[0, 1]$, $x \in [0, 1]$ ***then***

$$|\mathbf{B}_n^*(f; x) - f(x)| \le 2\omega(f, \sqrt{\delta_n})$$

where $\delta_n = \mathbf{B}_n^* .(t - x)^2; x\Sigma$

Proof : We have

$$|\mathbf{B}_n^*(f; x) - f(x)| \le \sum_{k=0}^{n} P_{n,k}(x).f\left(\frac{k}{n+1}\right) - f(x)\Sigma$$

$$\le \sum_{k=0}^{n} P_{n,k}(x)\omega_f\left(\left|\frac{k}{n+1} - x\right|\right)$$

$$\le \sum_{k=0}^{n} P_{n,k}(x)\left(1 + \frac{1}{\delta^*}\left|\frac{k}{n+1} - x\right|\Sigma^2\Sigma\right)\omega_f(\delta^*)$$

$$= \left(1 + \frac{1}{\delta^{*2}}\sum_{k=0}^{n} P_{n,k}(x)\left(\frac{k}{n+1} - x\right)^2\right)\omega_f(\delta^*)$$

$$= \Sigma 1 + \frac{1}{\delta^{*2}}\mathbf{B}^*_n.(t - x)^2; x\Sigma\Sigma\,\omega(\delta^*)$$

Now, if we choose

δ_{*2} $\mathbf{n}$

$$\delta^{*} = \delta_n = \mathbf{B}_n^{*}.(t-x)^2; x)\Sigma$$

then we obtain

$$|\mathbf{B}_n^{*}(f;x) - f(x)| \leq 2\omega(f, \sqrt{\delta_n})$$

Theorem 4.2. *If* $f \in Lip_C(r)$, $C > 0$, $0 < r \leq 1$ *then*

$$|\mathbf{B}_n^{*}(f;x) - f(x))| \leq C\delta_n^r(x)$$

where $\delta_n(x) = \mathbf{B}_n^{*}((t-x)^2; x)^{1/2}$

Proof: In view of the monotonicity and linearity of B_n^{*}, we get

$$|\mathbf{B}_n^{*}(f;x) - f(x)| \leq \sum_{k=0}^{n} P_{n,k}(x) . f(\frac{k}{n+1}) - f(x) . \leq C \sum_{k=0}^{n} P_{n,k}(x) . \frac{k}{n+1} - x .^{r}$$

$$= C \sum_{k=0}^{n} . P_{n,k}(x) . \frac{k}{n+1} - x \Sigma^{2}\Sigma^{r/2} (P_{n,k}(x))^{1-r/2}$$

Foom Holder's inequality, we obtain

$$= C . \sum_{k=0}^{n} P_{n,k}(x) \frac{k}{n+1} - x\Sigma_2\Sigma_{r/2} . \Sigma \sum_{k=0}^{n} P_{n,k}(x) \Sigma^{1-r/2}$$

$$= C . \sum_{k=0}^{n} P_{n,k}(x) . \frac{k}{n+1} - x\Sigma_2\Sigma_{r/2}$$

$$= C . \mathbf{B}_n^{*}(t-x)^2; x\Sigma\Sigma^{r/2}$$

$$= C\delta_n^r(x)$$

where $\delta_n(x) = \mathbf{B}_n^{*}((t-x)^2; x)^{1/2}$

5 Vorovnoskaja type theorem

Now, we give a *Vorovnoskaja type asymptotic formula* for our defined oper- ator B_n^{*}.

Lemma 5.1. *If* $x \in [0, 1]$, *then*

$$\lim_{n\to\infty} n.\mathbf{B}_n^{*}((t-x); x) = 1 - 3x$$

Proof : From (ii) of lemma 2.3 ,we have

$$\mathbf{B}_n^{*}((t-x); x)) = \frac{x(-3n-2)+n}{(n+1)(n+2)}$$

so we get

$$\lim_{n\to\infty} n.\mathbf{B}_n^{*}((t-x); x) = 1 - 3x$$

Lemma 5.2. *If* $x \in [0, 1]$, *then*

$$\lim_{n\to\infty} n.\mathbf{B}_n^{*}((t-x)^2; x) = x(7x + 2)$$

Proof : From (iii) of lemma 2.3 ,we have

$$\mathbf{B}_n^{*} .(t-x)^2; x\Sigma = \frac{x^2(7n^3+15n^2+17n+6)+x(2n^3-4n^2-2n)+(3n^2+n)}{(n+1)^2(n+2)(n+3)}$$

then we get

$$\lim_{n\to\infty} n.\mathbf{B}_n^*((t-x)^2;x) = x(7x+2)$$

Now,we prove the main theorem.

Theorem 5.1. *If* $f \in C[0,1]$ *and* $f', f'' \in C[0,1]$ *where* $x \in [0,1]$ *then*

$$\lim_{n\to\infty} n(\mathbf{B}_n^*(f;x) - f(x)) = (1-3x)f'(x) + x(\frac{7}{2}x+1)f''(x)$$

Proof : From Taylor's expansion,we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + h(t)(t-x)^2 \quad (4)$$

where $h(t) \in C[0,1]$ is Peano type remainder and $\lim_{t\to x} h(t) = 0$

Now

$$n(\mathbf{B}_n^*(f;x)-f(x)) = nf'(x)\mathbf{B}_n^*((t-x);x)+\frac{n}{2}f''(x)\mathbf{B}_n^*((t-x)^2;x)+n\mathbf{B}_n^*(h(t)(t-x)^2;x) \quad (5)$$

from Cauchy-Schwarz inequality, we get

$$n\mathbf{B}_n^*(h(t)(t-x)^2;x) \le \sqrt{\mathbf{B}_n(h^2(t);x)} \cdot \sqrt{n^2\mathbf{B}_n((t-x)^4;x)} \quad (6)$$

Here $n^2\mathbf{B}_n^*((t-x)^4;x)$ is nonnegative and finite for $x \in [0,1]$ refer to lemma 2.3 and $\lim_{t\to x} h(t) = 0$

$\mathbf{B}_n^*$ is uniformly convergent for $f(t) \in C[0,1]$ so we obtain

$$\lim_{n\to\infty} (h^2(t);x) = h^2(x) = 0$$

From (6) we have $n\mathbf{B}_n^*(h(t)(t-x)^2;x) = 0$

and from (5) of lemma 5.1,5.2

$$\lim_{n\to\infty} n(\mathbf{B}_n^*(f;x) - f(x)) = (1-3x)f'(x) + x(\frac{7}{2}x+1)f''(x)$$

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