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Unrestricted Gibonacci Quaternions

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Abstract:
Gibonacci sequence is a generalization of Fibonacci and Lucas sequences. In this study, we define unrestricted gibonacci quaternions by picking arbitrary elements of gibonacci sequence for the ordered basis of quaternions. After obtaining Binet formula for unrestricted gibonacci quaternions, we give generalizations of the some well – known identities..
***Key Word**: Gibonacci sequence; Quaternions; Binet formula.*

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I. Introduction

Fibonacci sequence $\{F_n\}$ and Lucas sequences $\{L_n\}$ are defined with numbers which satify the same recurrence relation, but the initial conditions, namely Fibonacci numbers satisfy

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 0$, $F_1 = 1$ and Lucas numbers satisfy

$$L_n = L_{n-1} + L_{n-2}$$

with initial conditions $L_0 = 2$, $L_1 = 1$. The well – known identities for Fibonacci and Lucas numbers are Binet formulas given in the following

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n$$

respectively. These two famous integer sequences have been studied by many authors because of their interesting properties. Also, there are many generalizations of the Fibonacci sequence. One of them is gibonacci sequence.

Gibonacci sequence is defined by the following recurrence relation

$$G_n = G_{n-1} + G_{n-2}$$

with initial conditions $G_1 = a$ and $G_2 = b$. One can easily see that setting $(a, b) \rightarrow (1,1)$ gives Fibonacci sequence whereas $(a, b) \rightarrow (1,3)$ gives Lucas sequence. The Binet formula for the gibonacci sequence is

$$G_n = \frac{c\alpha^n - d\beta^n}{\alpha - \beta} \quad (1)$$

where $c = a + (a - b)\beta$ and $d = a + (a - b)\alpha$. It should be noted that

$$cd = a^2 + ab - b^2. \quad (2)$$

The detailed knowledge for gibonacci sequence and its properties can be found in [10].

Quaternions were introduced by Sir W.R. Hamilton in 1843 in order to extend complex numbers. The set of all quaternions are shown as

$$\mathcal{H} = \{a + bi + cj + dk: a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}.$$

The set of $\{i, j, k\}$ satisfy the following relations

$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Conjugate of a quaternion $q = a + bi + cj + dk$ is

$$\bar{q} = a - bi - cj - dk$$

and the norm of q is

$$\|q\| = \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

The set of all quaternions $\mathcal{H}$ forms an associative normed division algebra over the real numbers with four-dimension.

Horadam [6] introduced Fibonacci quaternions as follows

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}.$$

Similarly, Iyer [8] defined Lucas quaternions in the following

$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}.$$

Halici [5] gave the following Binet formulas fort he Fibonacci and Lucas quaternions and this was a milestone fort his theory.

$$Q_n = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha-\beta} \text{ and } T_n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n.$$

By following her formulas, many studies have been carried out about Fibonacci quaternions or generalizations of Fibonacci quaternions so far [1, 2, 3, 4, 7, 9, 11, 12, 13, 14]. All these studies authors choose the consequtive Fibonacci numbers for the ordered basis $\{1, i, j, k\}$ of the quaternions. The innovation of this study is to choose random elements gibonacci sequence for the ordered basis $\{1, i, j, k\}$.

II. Generating Functions and Binet – Like Formulas

In this section, we define gibonacci quaternions and calculate some necessary properties of these quaternions including Binet formula.

Definition 2.1. For any non-negative integers x, y, z and n, the nth gibonacci quaternion is

$$G_n^{(x,y,z)} = G_n + G_{n+x}i + G_{n+y}j + G_{n+z}k. \tag{3}$$

By setting $(a, b) \to (1,1)$, Eq.(3) reduces the unrestricted Fibonacci quaternions:

$$F_n^{(x,y,z)} = F_n + F_{n+x}i + F_{n+y}j + F_{n+z}k \tag{4}$$

and setting $(a, b) \to (1,3)$ gives the unrestricted Lucas quaternions:

$$L_n^{(x,y,z)} = L_n + L_{n+x}i + L_{n+y}j + L_{n+z}k. \tag{5}$$

Definition 1 gives the following recurrence relation for the unrestricted gibonacci quaternions

$$G_n^{(x,y,z)} = G_{n-1}^{(x,y,z)} + G_{n-2}^{(x,y,z)}. \tag{6}$$

We know that $G_n = aF_{n-2} + bF_{n-1}$ from [10]. If this equation and the well-known equation $F_{-n} = (-1)^{n+1}F_n$ consider together, the relation $G_{-n} = (-1)^{n+1}(aF_{n+2} - bF_{n+1})$ can be obtained. So, we obtain unrestricted gibonacci quaternions for negative indices as follows

$$G_{-n}^{(x,y,z)} = (-1)^{n+1}\left[aF_{n+2}^{(-x,-y,-z)} - bF_{n+1}^{(-x,-y,-z)}\right]. \tag{7}$$

Binet formula for the unrestricted gibonacci quaternions is given in the following theorem.

Theorem 2.2. For any non-negative integers x, y, z and n, the nth gibonacci quaternion is

$$G_n^{(x,y,z)} = \frac{c\alpha^{(x,y,z)}\alpha^n - d\beta^{(x,y,z)}\beta^n}{\alpha-\beta} \tag{8}$$

where $\alpha^{(x,y,z)} = 1 + i\alpha^x + j\alpha^y + k\alpha^z$ and $\beta^{(x,y,z)} = 1 + i\beta^x + j\beta^y + k\beta^z$.

Proof. From Eqs. (1) and (3), we have

$$\begin{aligned} G_n^{(x,y,z)} &= G_n + G_{n+x}i + G_{n+y}j + G_{n+z}k \\ &= \frac{1}{\alpha-\beta}[c\alpha^n - d\beta^n + i(c\alpha^{n+x} - d\beta^{n+x}) + j(c\alpha^{n+y} - d\beta^{n+y}) + k(c\alpha^{n+z} - d\beta^{n+z})] \\ &= \frac{1}{\alpha-\beta}[c\alpha^n(1 + i\alpha^x + j\alpha^y + k\alpha^z) - d\beta^n(1 + i\beta^x + j\beta^y + k\beta^z)]. \end{aligned}$$

The last equation gives Eq.(8). □

We need the following properties for later calculations.

Lemma 2.3. For any integers x, y and z, we have

$$\alpha^{(x,y,z)}\beta^{(x,y,z)} = \theta^{(x,y,z)} + \sqrt{5}\mu^{(x,y,z)} \tag{9}$$

and

$$\beta^{(x,y,z)}\alpha^{(x,y,z)} = \theta^{(x,y,z)} - \sqrt{5}\mu^{(x,y,z)} \tag{10}$$

where

$$\theta^{(x,y,z)} = L_0^{(x,y,z)} - 1 - (-1)^x - (-1)^y - (-1)^z$$

and

$$\mu^{(x,y,z)} = i(-1)^z F_{y-z} + j(-1)^x F_{z-x} + k(-1)^y F_{x-y}.$$

Proof.From the product of two quaternions and the relation $\alpha\beta = -1$, we have

$$\begin{aligned} \alpha^{(x,y,z)}\beta^{(x,y,z)} &= (1 + i\alpha^x + j\alpha^y + k\alpha^z)(1 + i\beta^x + j\beta^y + k\beta^z) \\ &= 1 - (\alpha\beta)^x - (\alpha\beta)^y - (\alpha\beta)^z + i(\alpha^x + \beta^x + \alpha^y\beta^z - \alpha^z\beta^y) \\ &\quad + j(\alpha^y + \beta^y + \alpha^z\beta^x - \alpha^x\beta^z) + k(\alpha^z + \beta^z + \alpha^x\beta^y - \alpha^y\beta^x) \\ &= L_0^{(x,y,z)} - 1 - (-1)^x - (-1)^y - (-1)^z + i(-1)^z(\alpha^{y-z} - \beta^{y-z}) \\ &\quad + j(-1)^x(\alpha^{z-x} - \beta^{z-x}) + k(-1)^y(\alpha^{x-y} - \beta^{x-y}). \end{aligned}$$

We obtain Eq. (9) from the last equation. Eq.(10) can be obtained in a similar way.□

III. Results

By using Binet formula and Lemma 2.3, we can obtain many identities for the unrestricted gibonacci quaternions. We start with Vajda's identity given in the following.

Theorem 3.1. For any integers x, y, z, m, n and k, we have

$$G_{m+n}^{(x,y,z)}G_{m+k}^{(x,y,z)} - G_{m}^{(x,y,z)}G_{m+n+k}^{(x,y,z)} = (a^2+ab-b^2)(-1)^m F_n\left[\theta^{(x,y,z)}F_k - \mu^{(x,y,z)}L_k\right].$$

Proof. From the Binet formula (8), we have

$$\begin{aligned}
&G_{m+n}^{(x,y,z)}G_{m+k}^{(x,y,z)} - G_{m}^{(x,y,z)}G_{m+n+k}^{(x,y,z)}\\
&= \tfrac{1}{5}\big[\big(c\alpha^{(x,y,z)}\alpha^{m+n} - d\beta^{(x,y,z)}\beta^{m+n}\big)\big(c\alpha^{(x,y,z)}\alpha^{m+k} - d\beta^{(x,y,z)}\beta^{m+k}\big)\\
&\qquad -\big(c\alpha^{(x,y,z)}\alpha^{m} - d\beta^{(x,y,z)}\beta^{m}\big)\big(c\alpha^{(x,y,z)}\alpha^{m+n+k} - d\beta^{(x,y,z)}\beta^{m+n+k}\big)\big]\\
&= \tfrac{1}{5}\big[-cd\alpha^{(x,y,z)}\beta^{(x,y,z)}\alpha^{m+n}\beta^{m+k} - cd\beta^{(x,y,z)}\alpha^{(x,y,z)}\alpha^{m+k}\beta^{m+n}\\
&\qquad +cd\alpha^{(x,y,z)}\beta^{(x,y,z)}\alpha^{m}\beta^{m+n+k} + cd\beta^{(x,y,z)}\alpha^{(x,y,z)}\alpha^{m+n+k}\beta^{m}\big]\\
&= \tfrac{cd(-1)^m}{5}\big(-\alpha^{(x,y,z)}\beta^{(x,y,z)}\alpha^{n}\beta^{k} - \beta^{(x,y,z)}\alpha^{(x,y,z)}\alpha^{k}\beta^{n}\\
&\qquad +\alpha^{(x,y,z)}\beta^{(x,y,z)}\beta^{n+k} + \beta^{(x,y,z)}\alpha^{(x,y,z)}\alpha^{n+k}\big)\\
&= \tfrac{cd(-1)^m}{5}\big[\big(\theta^{(x,y,z)} + \sqrt{5}\mu^{(x,y,z)}\big)\beta^k(\beta^n - \alpha^n)\\
&\qquad +\big(\theta^{(x,y,z)} - \sqrt{5}\mu^{(x,y,z)}\big)\alpha^k(\alpha^n - \beta^n)\big] \quad \text{(From Eqs. (9) and (10))}\\
&= \tfrac{cd(-1)^m}{5}\big[\big(\sqrt{5}\theta^{(x,y,z)} + \mu^{(x,y,z)}\big)\beta^k(-F_n) + \big(\sqrt{5}\theta^{(x,y,z)} - \mu^{(x,y,z)}\big)\alpha^k F_n\big]\\
&= \tfrac{cd(-1)^m}{5}\big[\sqrt{5}\theta^{(x,y,z)}F_n(\alpha^k - \beta^k) - \mu^{(x,y,z)}F_n(\alpha^k + \beta^k)\big].
\end{aligned}$$

The last equation and Eq.(2) prove the theorem.□

The Vajda's identity is a generalization of Catalan's identity. If we take $k \to (-n)$ in Vajda's identity, we obtain the Catalan's identity given in the next theorem.

Theorem 3.2. For any integers x, y, z, m and n, we have

$$G_{m+n}^{(x,y,z)}G_{m-n}^{(x,y,z)} - \left[G_{m}^{(x,y,z)}\right]^2 = (a^2+ab-b^2)(-1)^{m+1}F_n\left[\theta^{(x,y,z)}F_n + \mu^{(x,y,z)}L_n\right].$$

If we substitute $n \to 1$ into the Catalan's identity, we obtain Cassini's identity given in the following theorem.

Theorem 3.3. For any integers x, y, z and m, we have

$$G_{m+1}^{(x,y,z)}G_{m-1}^{(x,y,z)} - \left[G_{m}^{(x,y,z)}\right]^2 = (a^2+ab-b^2)(-1)^{m+1}\left[\theta^{(x,y,z)} + \mu^{(x,y,z)}\right].$$

Another well – known identity is d'Ocagne's identity. We give it in the next theorem.

Theorem 3.4. For any integers x, y, z, m and n, we have

$$G_{m}^{(x,y,z)}G_{n+1}^{(x,y,z)} - G_{m+1}^{(x,y,z)}G_{n}^{(x,y,z)} = (a^2+ab-b^2)(-1)^n\left[\theta^{(x,y,z)}F_{m-n} + \mu^{(x,y,z)}L_{m-n}\right].$$

Proof. From the Binet formula for the unrestricted gibonacci quaternions, we have

$$\begin{aligned}
&G_{m}^{(x,y,z)}G_{n+1}^{(x,y,z)} - G_{m+1}^{(x,y,z)}G_{n}^{(x,y,z)}\\
&= \tfrac{1}{5}\big[\big(c\alpha^{(x,y,z)}\alpha^{m} - d\beta^{(x,y,z)}\beta^{m}\big)\big(c\alpha^{(x,y,z)}\alpha^{n+1} - d\beta^{(x,y,z)}\beta^{n+1}\big)\\
&\qquad -\big(c\alpha^{(x,y,z)}\alpha^{m+1} - d\beta^{(x,y,z)}\beta^{m+1}\big)\big(c\alpha^{(x,y,z)}\alpha^{n} - d\beta^{(x,y,z)}\beta^{n}\big)\big]\\
&= \tfrac{cd}{5}\big[-\alpha^{(x,y,z)}\beta^{(x,y,z)}\alpha^{m}\beta^{n+1} - \beta^{(x,y,z)}\alpha^{(x,y,z)}\alpha^{n+1}\beta^{m}\\
&\qquad +\alpha^{(x,y,z)}\beta^{(x,y,z)}\alpha^{m+1}\beta^{n} + \beta^{(x,y,z)}\alpha^{(x,y,z)}\alpha^{n}\beta^{m+1}\big]\\
&= \tfrac{cd(-1)^n}{5}\big[\big(\theta^{(x,y,z)} + \sqrt{5}\mu^{(x,y,z)}\big)\alpha^{m-n}(\alpha-\beta) - \big(\theta^{(x,y,z)} - \sqrt{5}\mu^{(x,y,z)}\big)\beta^{m-n}(\alpha-\beta)\big]\\
&= \tfrac{cd(-1)^n}{5}\big[\sqrt{5}\theta^{(x,y,z)}(\alpha^{m-n} - \beta^{m-n}) - 5\mu^{(x,y,z)}(\alpha^{m-n} + \beta^{m-n})\big]\\
&= cd(-1)^n\big[\theta^{(x,y,z)}F_{m-n} + \mu^{(x,y,z)}L_{m-n}\big].
\end{aligned}$$

The last equation proves the theorem.□

We obtain some identities for unrestricted gibonacci quaternions and give in the following theorem without proofs.

Theorem 3.5.

$$G_{n}^{(x,y,z)} = G_{n-1}^{(x,y,z)} + G_{n-2}^{(x,y,z)},$$

$$\sum_{k=1}^{m} G_{n+k}^{(x,y,z)} = G_{m+n+2}^{(x,y,z)} - G_{n+2}^{(x,y,z)},$$

$$\sum_{k=1}^{m} G_k^{(x,y,z)} = G_{m+2}^{(x,y,z)} - b,$$
$$G_{m+n}^{(x,y,z)} = G_m^{(x,y,z)} F_{n+1} + G_{m-1}^{(x,y,z)} F_n,$$
$$G_{m-n}^{(x,y,z)} = (-1)^n \left[G_m^{(x,y,z)} F_{n-1} - G_{m-1}^{(x,y,z)} F_n \right].$$

IV. Conclusion

There are many generalizations for Fibonacci and Lucas sequences and some studies on Fibonacci and Lucas quaternions. Some authors studied Hamilton quaternions whose coefficients are classical Fibonacci and Lucas numbers or their generalizations, whereas some authors studied other quaternion systems other than Hamilton quaternions whose coefficients Fibonacci and Lucas numbers also. This study focused on Hamilton quaternions whose coefficients are gibonacci numbers, another generalization of Fibonacci and Lucas numbers. The difference between other studies and the current one is to be able to choose arbitrary gibonacci numbers for the coefficients of the ordered basis $\{1, i, j, k\}$ of the Hamilton quaternions. Therefore, we named these quaternions as unrestricted gibonacci quaternions. We gave Binet formula for these quaternions and many interesting identities for them.

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