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Discrete-Time Linear Stochastic Model for the Analysis of Production Systems Functioning in an Unstable External Environment

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Abstract:*In the given paper some linear discrete-time stochastic model intended for simulation the behavior of an analyzed production system functioning in an unstable external environment is proposed.In this model destabilizing effects on the inventory and on the external supplies of the analyzed production system are presented explicitly.The average and the second order behaviors of the proposed model is analyzed.Some two-dimensional time-invariant production system is provided, as an example.The step-by-step adaptive approach for correction the planned standard behavior of the production system functioning in the unstable external environment is proposed.This approach is based on the assumption that the estimates of the returned or non-conforming inventory and of unforeseen changes in the deliveries of the inventory are true.*

Key Word*: Production systems, planning, unstable environment, discrete-time, linear stochastic model,inventory control.*

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I. Introduction

It is well-known, that efficient production planning is one of the key problems for production systems surviving.To solve this problem, various models, both deterministic [1,2] and stochastic [3-5] have been used.

The stochastic Markov model $X_{k+1} = f_{\theta_{k+1}}(X_k)\ (k \in \mathbf{Z}_+)$, where $\theta_{k+1}\ (k \in \mathbf{Z}_+)$ are independent drawings from a given probability distribution μ, and $\{f_\theta : \mathbf{R}^n \to \mathbf{R}^n \mid \theta \in \Theta\}$ is a given set of mappingshad been investigated in detail in [6].Unfortunately, it is difficult to use these results for solving specific applied problems.Possibly, this wasthe main reason why considerable attention has been focused on the study of linear stochastic models [7, 8].There have been thoroughly investigated the models $X_{k+1} = A_k X_k + B_k\ (k \in \mathbf{Z}_+)$ [9-12] and $\mathrm{E}(X_{k+1}) = A_k X_k + B_k\ (k \in \mathbf{Z}_+)$ [13], where $X_k\ (k \in \mathbf{Z}_+)$ are n-dimensional real-valued random vectors and A_k, $B_k\ (k \in \mathbf{Z}_+)$ are real-valued (in general case, random) $n \times n$-matrices.Common to both models is that they do not represent explicitly the external deliveries to the production system.This drawback has been eliminated in the following model [13, 14].

Let n-dimensional random variable x_0, m-dimensional deterministic time sequence $u_k\ (k \in \mathbf{Z}_+)$, and l-dimensional stochastic process $w_k\ (k \in \mathbf{Z}_+)$ be given.A linear discrete-time stochastic model of production planning is defined by the recurrence relation

$$x_{k+1} = A_k x_k + B_k u_k + G_k w_k \quad (k \in \mathbf{Z}_+), \tag{1}$$

where A_k, B_k and G_k are given matrices of dimensions $n \times n$, $n \times m$ and $n \times l$, respectively,for all $(k \in \mathbf{Z}_+)$.

*Remark 1.*The model (1) can be interpreted as follows. Let x_k be the store inventory level at the beginning of the day k, u_k be the inventory ordered and delivered at the beginning of the day k, and w_k be the random amount of inventory sold during the day k.Then x_{k+1} is the store inventory level at the beginning of the day $k+1$.

It is natural to assume that the model (1) represents the planned standard behavior of the analyzed production system.However, there is the problem to design this model when the analyzed production system is functioning in an unstable external environment.In the given paper the following adaptive approach to solve this problem is proposed.Some generalization of the model (1) that presents in the explicit form the destabilizing actions of the external environment is defined and analyzed.Based on the simulation of the current version of the model (1) and the proposed model, deviations in their behavior can be calculated. If these deviations exceed the given threshold, then the model (1) is corrected.

The rest of the paper is organized as follows.In Section 2 the well-known approach for analysis of the model (1) is presented briefly.In Section 3 proposed generalization of the model (1) intended for simulation the behavior of the analyzed production system in the unstable external environment is defined.In Section4 the behavior of the proposed model is investigated.In Section 5 the step-by-step adaptive approach intended to correct the model (1) when the analyzed production system is functioning in an unstable external environment is proposed. Section 6 contains concluding remarks.

II. Brief analysis of the model (1)

Usually, the analysis of the model (1) is carried as follows (see \cite{15}, for example).
Firstly, the solution of the recurrence relation (1) is found:

$$x_k = \Phi(k,0)x_0 + \sum_{j=0}^{k-1}\Phi(k, j+1)(B_j u_j + G_j w_j) \quad (k \in \mathbf{Z}_+), \tag{2}$$

where for all $j \le k$

$$\Phi(k, j) = \begin{cases} A_{k-1}\dots A_j, \ if\ j = 0,1,\dots,k-1 \\ I_n, \ if\ j = k \end{cases} \quad (k \in \mathbf{Z}_+),$$

and I_n is the identity $n \times n$-matrix.

When analyzing the average behavior of the model (1), usually it is assumed that w_k $(k \in \mathbf{Z}_+)$ is an independent sequence, it is also independent of x_0, and $\mathrm{E}(w_k) = 0_l$ $(k \in \mathbf{Z}_+)$, where 0_l is the zero l-dimensional vector.
Hence, x_k is a Markov process.
It follows from (1) and (2) that, respectively

$$\mathrm{E}(x_{k+1}) = A_k \mathrm{E}(x_k) + B_k u_k \quad (k \in \mathbf{Z}_+), \tag{3}$$

and

$$\mathrm{E}(x_k) = \Phi(k,0)\mathrm{E}(x_0) + \sum_{j=0}^{k-1}\Phi(k, j+1)B_j u_j \quad (k \in \mathbf{Z}_+). \tag{4}$$

To analyze the second order properties of the model (1), it is assumed additionally that $\mathrm{E}(w_k w_j^T) = Q_k \delta_{kj}$ $(k, j \in \mathbf{Z}_+)$, where δ_{kj} is the Kronecker delta function (and, thus, w_k $(k \in \mathbf{Z}_+)$ is the wide-sense white noise), and $\mathrm{E}(x_0 w_j^T) = O_{n,l}$, where $O_{n,l}$ isthe zero $n \times l$-matrix.
Let $\tilde{x}_k = x_k - \mathrm{E}(x_k)$ $(k \in \mathbf{Z}_+)$. It follows from (1), (3) and (2), (4) that, respectively

$$\tilde{x}_{k+1} = A_k \tilde{x}_k + G_k w_k \quad (k \in \mathbf{Z}_+), \tag{5}$$

and

$$\tilde{x}_k = \Phi(k,0)\tilde{x}_0 + \sum_{j=0}^{k-1}\Phi(k, j+1)G_j w_j \quad (k \in \mathbf{Z}_+). \tag{6}$$

Remark 2.Since $\tilde{x}_k$ $(k \in \mathbf{Z}_+)$ depends linearly on $\tilde{x}_0$ and $w_0,\dots,w_{k-1}$, then $\mathrm{E}(\tilde{x}_k w_k^T) = O_{n,l}$

The covariance matrix $\Sigma_{x_k} = \mathrm{E}(\tilde{x}_k \tilde{x}_k^T)$ $(k \in \mathbf{Z}_+)$ can be computed from (5) and (6), as follows:

$$\Sigma_{x_{k+1}} = A_k \Sigma_{x_k} A_k^T + G_k Q_k G_k^T \quad (k \in \mathbf{Z}_+),$$

and

$$\Sigma_{x_k} = \Phi(k,0)\Sigma_{x_0}\Phi^T(k,0) + \sum_{j=0}^{k-1}\Phi(k, j+1)G_j Q_j G_j^T \Phi^T(k, j+1) \quad (k \in \mathbf{Z}_+). \tag{7}$$

To analyze the asymptotic behavior of $\mathrm{E}(x_k)$ and Σ_{x_k} (i.e. the behavior when $k \to \infty$), it is assumed additionally that $\|\Phi(k,0)\| \le c\rho^k$ $(k \in \mathbf{Z}_+)$ for some positive constants c and $\rho < 1$.Therefore, it follows from (4) and (7) that, respectively

$$\lim_{k\to\infty}\mathrm{E}(x_k) = \lim_{k\to\infty}\sum_{j=0}^{k-1}\Phi(k, j+1)B_j u_j, \tag{8}$$

and

$$\lim_{k\to\infty}\Sigma_{x_k} = \lim_{k\to\infty}\sum_{j=0}^{k-1}\Phi(k, j+1)G_j Q_j G_j^T \Phi^T(k, j+1) . \qquad (9)$$

The following two theorems directly follow from the equalities (8) and (9).

Theorem 1. *If* $\|\Phi(k,0)\| \le c\rho^k$ $(k \in \mathbf{Z}_+)$ *for some positive constants* c *and* $\rho < 1$, *then the influence of the mean of the random initial condition* x_0 *vanishes, when* $k \to \infty$.

Theorem 2. *Let* $\|\Phi(k, j)\| \le c\rho^{k-j}$ $(j = 0,1,\ldots,k)$, $\|G_k\| \le c$, $\|Q_k\| \le c$, $\|B_k\| \le c$ *and* $\|u_k\| \le c$ *for all* $k \in \mathbf{Z}_+$, *where* c *and* $\rho < 1$ *are some positive constants. Then* $\lim_{k\to\infty}\mathrm{E}(x_k)$ *and* $\lim_{k\to\infty}\Sigma_{x_k}$ *are finite limits both.*

As a rule, the behavior of the model (1) is characterized by the equalities (2), (4) and (7)-(9).
Let us illustrate the application of the above technique by the following example.

Example 1. Let us consider the 2-dimensional time-invariant model (1), such that $G_k = Q_k = I_2$, $A_k = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$, and $B_k = \begin{bmatrix} e_3 & 0 \\ 0 & e_4 \end{bmatrix}$ for all $k \in \mathbf{Z}_+$, where $e_i < 1$ $(i = 1,\ldots,4)$ are some positive constants, under assumption that $x_0^{(1)}$ and $x_0^{(2)}$ are independent random variables.

From (2), (4), and (7)-(9) we get, respectively,

$$\begin{cases} x_k^{(1)} = e_1^k x_0^{(1)} + \sum_{j=0}^{k-1} e_1^{k-1-j}(e_3 u_j^{(1)} + w_j^{(1)}) \\ x_k^{(2)} = e_2^k x_0^{(2)} + \sum_{j=0}^{k-1} e_2^{k-1-j}(e_4 u_j^{(2)} + w_j^{(2)}) \end{cases} \quad (k \in \mathbf{Z}_+) ,$$

$$\begin{cases} \mathrm{E}(x_k^{(1)}) = e_1^k \mathrm{E}(x_0^{(1)}) + e_3 \sum_{j=0}^{k-1} e_1^{k-1-j} u_j^{(1)} \\ \mathrm{E}(x_k^{(2)}) = e_2^k \mathrm{E}(x_0^{(2)}) + e_4 \sum_{j=0}^{k-1} e_2^{k-1-j} u_j^{(2)} \end{cases} \quad (k \in \mathbf{Z}_+) ,$$

$$\Sigma_{x_k} = \begin{bmatrix} e_1^{2k}\sigma^2(x_0^{(1)}) + \dfrac{1-e_1^{2k}}{1-e_1^2} & 0 \\ 0 & e_2^{2k}\sigma^2(x_0^{(2)}) + \dfrac{1-e_2^{2k}}{1-e_2^2} \end{bmatrix} \quad (k \in \mathbf{Z}_+) ,$$

$$\begin{cases} \lim_{k\to\infty}\mathrm{E}(x_k^{(1)}) = e_3 \lim_{k\to\infty}\sum_{j=0}^{k-1} e_1^{k-1-j} u_j^{(1)} \\ \lim_{k\to\infty}\mathrm{E}(x_k^{(2)}) = e_4 \lim_{k\to\infty}\sum_{j=0}^{k-1} e_2^{k-1-j} u_j^{(2)} \end{cases} ,$$

$$\lim_{k\to\infty}\Sigma_{x_k} = \begin{bmatrix} \dfrac{1-e_1^{2k}}{1-e_1^2} & 0 \\ 0 & \dfrac{1-e_2^{2k}}{1-e_2^2} \end{bmatrix} .$$

In what follows, the model (1) is interpreted as the planned standard behavior of the analyzed production system.

III. Proposed model of a production system

A natural generalization of the model (1) intended for simulation the behavior of the analyzed production system in the unstable external environment can be defined by the recurrence relation

$$y_{k+1} = \alpha_k A_k y_k + \beta_k B_k u_k + G_k w_k \quad (k \in \mathbf{Z}_+) , \qquad (10)$$

where α_k $(k \in \mathbf{Z}_+)$ and β_k $(k \in \mathbf{Z}_+)$ are given 1-dimensional stochastic processes, such that $\mathrm{E}(\alpha_k) = a$ $(k \in \mathbf{Z}_+)$ and $\mathrm{E}(\beta_k) = b$ $(k \in \mathbf{Z}_+)$ for some given positive numbers a and b, and $y_0 = x_0$.

It is evident that the recurrence relation (10) is transformed into the recurrence relation (1), if $\alpha_k = \beta_k = 1$ for all $k \in \mathbf{Z}_+$, i.e. if the external environment is stable.

*Remark 3.*In the context of Remark 1, the process α_k $(k \in \mathbf{Z}_+)$ characterizes the amount of returned or non-conforming inventory during the day k, while the process β_k $(k \in \mathbf{Z}_+)$ characterizes unforeseen changes in the deliveries of the inventory at the beginning of the day k.

Solving the recurrence relation (10), we obtain that its solution has the following form:

$$y_k = \left(\prod_{i=0}^{k-1}\alpha_i\right)\Phi(k,0)x_0 + \sum_{j=0}^{k-1}\left(\prod_{h=j+1}^{k-1}\alpha_i\right)\Phi(k, j+1)(\beta_j B_j u_j + G_j w_j) \quad (k \in \mathbf{Z}_+) . \tag{11}$$

Let $z_k = y_k - x_k$ $(k \in \mathbf{Z}_+)$. From (1), (10) and (2), (11) we get, respectively,

$$z_{k+1} = \alpha_k A_k z_k + (\alpha_k - 1)A_k x_k + (\beta_k - 1)B_k u_k \quad (k \in \mathbf{Z}_+) , \tag{12}$$

and

$$z_k = \left(\prod_{i=0}^{k-1}\alpha_i - 1\right)\Phi(k,0)x_0 + \sum_{j=0}^{k-1}\left(\left(\beta_j\prod_{h=j+1}^{k-1}\alpha_h - 1\right)\Phi(k, j+1)B_j u_j + \left(\prod_{h=j+1}^{k-1}\alpha_h - 1\right)\Phi(k, j+1)G_j w_j\right) \quad (k \in \mathbf{Z}_+) . \tag{13}$$

Remark 4. The random variable $z_k = y_k - x_k$ $(k \in \mathbf{Z}_+)$ characterizes deviation of the analyzed production system behavior in the unstable environment from its planed standard behavior in the day k.

IV. Analysis of the proposed modelof a production system

To analyze the average behavior of the model (10), it is assumed that all assumptions of the previous Sections hold.Besides, it is assumed that each of the sequences α_k $(k \in \mathbf{Z}_+)$ and β_k $(k \in \mathbf{Z}_+)$ is independent, the stochastic processes α_k $(k \in \mathbf{Z}_+)$ and β_k $(k \in \mathbf{Z}_+)$ are independent, and each of them is independent of the stochastic process w_k $(k \in \mathbf{Z}_+)$ and of x_0 both.From (10) and (11), we get, respectively

$$\mathrm{E}(y_{k+1}) = aA_k\mathrm{E}(y_k) + bB_k u_k \quad (k \in \mathbf{Z}_+) , \tag{14}$$

and

$$\mathrm{E}(y_k) = a^k\Phi(k,0)\mathrm{E}(x_0) + b\sum_{j=0}^{k-1}a^{k-1-j}\Phi(k, j+1)B_j u_j \quad (k \in \mathbf{Z}_+) . \tag{15}$$

Similarly, from (12) and (13) we get, respectively

$$\mathrm{E}(z_{k+1}) = aA_k\mathrm{E}(z_k) + (a-1)A_k\mathrm{E}(x_k) + (b-1)B_k u_k \quad (k \in \mathbf{Z}_+) , \tag{16}$$

and

$$\mathrm{E}(z_k) = (a^k - 1)\Phi(k,0)\mathrm{E}(x_0) + \sum_{j=0}^{k-1}(ba^{k-1-j} - 1)\Phi(k, j+1)B_j u_j \quad (k \in \mathbf{Z}_+) . \tag{17}$$

Definition.*The model (10) is an exact in average representation of the model (1) if the equalities* $\mathrm{E}(z_k) = 0_n$ $(k \in \mathbf{Z}_+)$ *hold for any* x_0 *and* u_k $(k \in \mathbf{Z}_+)$.

The following lemma directly follows from the equality (16).

Lemma.*The equalities* $\mathrm{E}(z_k) = 0_n$ $(k \in \mathbf{Z}_+)$ *hold for any* x_0 *and* u_k $(k \in \mathbf{Z}_+)$ *if and only if the equalities* $a = b = 1$ *hold.*

In what follows, it is assumed that the model (10) is an exact in average representation of the model (1).
Due to this assumption, we get

$$\mathrm{E}(y_k) = \mathrm{E}(x_k) = \Phi(k,0)\mathrm{E}(x_0) + \sum_{j=0}^{k-1}\Phi(k, j+1)B_j u_j \quad (k \in \mathbf{Z}_+) , \tag{18}$$

and

$$\lim_{k\to\infty}\mathrm{E}(y_k) = \lim_{k\to\infty}\mathrm{E}(x_k) = \lim_{k\to\infty}\sum_{j=0}^{k-1}\Phi(k, j+1)B_j u_j . \tag{19}$$

Now we analyze the second order properties of the model (10).
Let $\tilde{y}_k = y_k - \mathrm{E}(y_k)$ $(k \in \mathbf{Z}_+)$. From (10), (14) and (11), (15) we get, respectively,

$$\tilde{y}_{k+1} = \alpha_k A_k \tilde{y}_k + G_k w_k + \gamma_k \quad (k \in \mathbf{Z}_+) , \tag{20}$$

where

$$\gamma_k = (\alpha_k - 1)A_k \mathrm{E}(x_k) + (\beta_k - 1)B_k u_k \quad (k \in \mathbf{Z}_+), \tag{21}$$

and

$$\tilde{y}_k = \left(\prod_{i=0}^{k-1}\alpha_i\right)\Phi(k,0)\tilde{x}_0 + \sum_{j=0}^{k-1}\left(\beta_j \prod_{h=j+1}^{k-1}\alpha_h - 1\right)\Phi(k, j+1)B_j u_j +$$
$$+\sum_{j=0}^{k-1}\left(\prod_{h=j+1}^{k-1}\alpha_h\right)\Phi(k, j+1)G_j w_j + \left(\prod_{i=0}^{k-1}\alpha_i - 1\right)\Phi(k,0)\mathrm{E}(x_0) \quad (k \in \mathbf{Z}_+). \tag{22}$$

From (20) and (21) we get

$$\Sigma_{y_{k+1}} = \mathrm{E}(\alpha_k^2)A_k\Sigma_{y_k}A_k^T + G_kQ_kG_k^T + \sigma^2(\alpha_k)A_k\mathrm{E}(x_k)\mathrm{E}^T(x_k)A_k^T + \sigma^2(\beta_k)B_ku_ku_k^TB_k^T \quad (k \in \mathbf{Z}_+).$$

Similarly, from (22) we get

$$\Sigma_{y_k} = \mathrm{E}\left(\prod_{i=0}^{k-1}\alpha_i^2\right)\Phi(k,0)\Sigma_{x_0}\Phi^T(k,0) + \sum_{j=0}^{k-1}\sum_{r=0}^{k-1}\mathrm{E}(\varsigma_{jr})\Phi(k, j+1)B_ju_ju_r^TB_r^T\Phi^T(k, r+1) +$$
$$+\sum_{j=0}^{k-1}\mathrm{E}(\eta_j)(\Phi(k, j+1)B_ju_j\mathrm{E}^T(x_0)\Phi^T(k,0) + \Phi(k,0)\mathrm{E}(x_0)u_j^TB_j^T\Phi^T(k, j+1)) +$$
$$+\sum_{j=0}^{k-1}\mathrm{E}\left(\prod_{h=j+1}^{k-1}\alpha_h^2\right)\Phi(k, j+1)G_jQ_kG_j^T\Phi^T(k, j+1) + (\mathrm{E}\left(\prod_{i=0}^{k-1}\alpha_i^2\right) - 1)\Phi(k,0)\mathrm{E}(x_0)\mathrm{E}^T(x_0)\Phi^T(k,0) \quad (k \in \mathbf{Z}_+), \tag{23}$$

where

$$\varsigma_{jr} = (\beta_j\prod_{h=j+1}^{k-1}\alpha_h - 1)(\beta_r\prod_{s=r+1}^{k-1}\alpha_s - 1) \quad (j, r = 1,\ldots,k-1),$$

and

$$\eta_j = (\beta_j\prod_{h=j+1}^{k-1}\alpha_h - 1)(\prod_{i=0}^{k-1}\alpha_i - 1) \quad (j = 1,\ldots,k-1).$$

To analyze the asymptotic behavior of Σ_{y_k}, it is assumed that $\|\Phi(k,0)\| \le c\rho^k$ $(k \in \mathbf{Z}_+)$ and $\mathrm{E}\left(\prod_{i=0}^{k-1}\alpha_i^2\right) \le c\varepsilon^{-2k}$ $(k \in \mathbf{Z}_+)$ for some positive constants c, ρ $(\rho < 1)$ and ε $(\varepsilon > \rho)$. From (23) we get

$$\lim_{k\to\infty}\Sigma_{y_k} = \lim_{k\to\infty}\sum_{j=0}^{k-1}\sum_{r=0}^{k-1}\mathrm{E}(\varsigma_{jr})\Phi(k, j+1)B_ju_ju_r^TB_r^T\Phi^T(k, r+1) + \lim_{k\to\infty}\sum_{j=0}^{k-1}\mathrm{E}(\eta_j)(\Phi(k, j+1)B_ju_j\mathrm{E}^T(x_0)\Phi(k,0) +$$
$$+\Phi(k,0)\mathrm{E}(x_0)u_j^TB_j^T\Phi^T(k, j+1)) + \lim_{k\to\infty}\sum_{j=0}^{k-1}\mathrm{E}\left(\prod_{h=j+1}^{k-1}\alpha_h^2\right)\Phi(k, j+1)G_jQ_kG_j^T\Phi^T(k, j+1). \tag{24}$$

From (19), Theorem 1 and (23) we get the following theorem.

Theorem 3. *Let* $\|\Phi(k,0)\| \le c\rho^k$ *and* $\mathrm{E}\left(\prod_{i=0}^{k-1}\alpha_i^2\right) \le c\varepsilon^{-2k}$ *for all* $k \in \mathbf{Z}_+$, *where* c $(c > 0)$, ρ $(\rho < 1)$ *and* ε $(\varepsilon > \rho)$ *are some constants. Then the influence of the mean of the random initial condition* x_0 *vanishes, when* $k \to \infty$.

Besides, from (19) and (24) we get the following theorem.

Theorem 4. *Let for all* $k \in \mathbf{Z}_+$ *hold the inequalities* $\|\Phi(k, j)\| \le c\rho^{k-j}$ $(j = 0,1,\ldots,k)$, $\|G_k\| \le c$, $\|Q_k\| \le c$, $\|B_k\| \le c$, $\|u_k\| \le c$, $\mathrm{E}\left(\prod_{i=0}^{k-1}\alpha_i^2\right) \le c\varepsilon^{-2k}$, $|\mathrm{E}(\varsigma_{jr})| \le c\varepsilon^{-(2k-j-r)}$ $(j, r = 1,\ldots,k-1)$, $|\mathrm{E}(\varsigma_j)| \le c\varepsilon^{-(2k-j)}$ $(j = 1,\ldots,k-1)$, *and* $\mathrm{E}\left(\prod_{h=j+1}^{k-1}\alpha_h^2\right) \le c\varepsilon^{-2(k-j)}$ $(j = 0,1,\ldots,k-1)$, *where* c $(c > 0)$, ρ $(\rho < 1)$ *and* ε $(\varepsilon > \rho)$ *are some constants. Then* $\lim_{k\to\infty}\mathrm{E}(y_k)$ *and* $\lim_{k\to\infty}\Sigma_{y_k}$ *are finite limits both.*

Example 2. Let us consider the 2 -dimensional model (10) under the assumptions of Example 1. From (11), (18), (19), (23) and (24) we get:

$$\begin{cases} y_k^{(1)} = \left(\prod_{i=0}^{k-1}\alpha_i\right) e_1^k x_0^{(1)} + \sum_{j=0}^{k-1}\left(\prod_{h=j+1}^{k-1}\alpha_h\right) e_1^{k-1-j}(e_3\beta_j u_j^{(1)} + w_j^{(1)}) \\ y_k^{(2)} = \left(\prod_{i=0}^{k-1}\alpha_i\right) e_2^k x_0^{(2)} + \sum_{j=0}^{k-1}\left(\prod_{h=j+1}^{k-1}\alpha_h\right) e_2^{k-1-j}(e_4\beta_j u_j^{(2)} + w_j^{(2)}) \end{cases} \quad (k \in \mathbf{Z}_+),$$

$$\begin{cases} \mathrm{E}(y_k^{(1)}) = e_1^k \mathrm{E}(x_0^{(1)}) + e_3 \sum_{j=0}^{k-1} e_1^{k-1-j} u_j^{(1)} \\ \mathrm{E}(y_k^{(2)}) = e_2^k \mathrm{E}(x_0^{(2)}) + e_4 \sum_{j=0}^{k-1} e_2^{k-1-j} u_j^{(2)} \end{cases} \quad (k \in \mathbf{Z}_+),$$

$$\Sigma_{y_k} = \mathrm{E}\left(\prod_{i=0}^{k-1}\alpha_i^2\right)\begin{bmatrix} e_1^{2k}\sigma^2(x_0^{(1)}) & 0 \\ 0 & e_2^{2k}\sigma^2(x_0^{(2)}) \end{bmatrix} + \sum_{j=0}^{k-1}\sum_{r=0}^{k-1}\mathrm{E}(\varsigma_{jr})M_{jr}^{(1)} + \sum_{j=0}^{k-1}\mathrm{E}(\eta_j)(M_j^{(2)} + M_j^{(3)}) +$$
$$+\sum_{j=0}^{k-1}\mathrm{E}\left(\prod_{h=j+1}^{k-1}\alpha_h^2\right)\begin{bmatrix} e_1^{2(k-1-j)} & 0 \\ 0 & e_2^{2(k-1-j)} \end{bmatrix} + \left(\mathrm{E}\left(\prod_{i=0}^{k-1}\alpha_i^2\right) - 1\right)\begin{bmatrix} e_1^{2k}\mathrm{E}^2(x_0^{(1)}) & (e_1e_2)^k\mathrm{E}(x_0^{(1)})\mathrm{E}(x_0^{(2)}) \\ (e_1e_2)^k\mathrm{E}(x_0^{(1)})\mathrm{E}(x_0^{(2)}) & e_2^{2k}\mathrm{E}^2(x_0^{(2)}) \end{bmatrix} \quad (k \in \mathbf{Z}_+),$$

$$\begin{cases} \lim_{k\to\infty}\mathrm{E}(y_k^{(1)}) = e_3 \lim_{k\to\infty}\sum_{j=0}^{k-1} e_1^{k-1-j}u_j^{(1)} \\ \lim_{k\to\infty}\mathrm{E}(y_k^{(2)}) = e_4 \lim_{k\to\infty}\sum_{j=0}^{k-1} e_2^{k-1-j}u_j^{(2)} \end{cases},$$

$$\lim_{k\to\infty}\Sigma_{y_k} = \lim_{k\to\infty}\sum_{j=0}^{k-1}\sum_{r=0}^{k-1}\mathrm{E}(\varsigma_{jr})M_{jr}^{(1)} + \lim_{k\to\infty}\sum_{j=0}^{k-1}\mathrm{E}(\eta_j)(M_j^{(2)} + M_j^{(3)}) + \lim_{k\to\infty}\sum_{j=0}^{k-1}\mathrm{E}\left(\prod_{h=j+1}^{k-1}\alpha_h^{(2)}\right)\begin{bmatrix} e_1^{2(k-1-j)} & 0 \\ 0 & e_2^{2(k-1-j)} \end{bmatrix},$$

where

$$M_{jr}^{(1)} = \begin{bmatrix} e_3^2 e_1^{2k-1-j-r} u_j^{(1)} u_r^{(1)} & e_3 e_4 e_1^{k-1-j} e_2^{k-1-r} u_j^{(1)} u_r^{(2)} \\ e_3 e_4 e_1^{k-1-r} e_2^{k-1-j} u_j^{(2)} u_r^{(1)} & e_4^2 e_2^{2k-1-j-r} u_j^{(2)} u_r^{(2)} \end{bmatrix},$$

$$M_j^{(2)} = \begin{bmatrix} e_3 e_1^{2k-1-j}\mathrm{E}(x_0^{(1)})u_j^{(1)} & e_3 e_1^{k-1-j} e_2^k \mathrm{E}(x_0^{(2)})u_j^{(1)} \\ e_4 e_1^k e_2^{k-1-j}\mathrm{E}(x_0^{(1)})u_j^{(2)} & e_4 e_2^{2k-1-j}\mathrm{E}(x_0^{(2)})u_j^{(2)} \end{bmatrix},$$

and

$$M_j^{(3)} = \begin{bmatrix} e_3 e_1^{2k-1-j}\mathrm{E}(x_0^{(1)})u_j^{(1)} & e_4 e_1^k e_2^{k-1-j}\mathrm{E}(x_0^{(1)})u_j^{(2)} \\ e_3 e_1^{k-1-j} e_2^k \mathrm{E}(x_0^{(2)})u_j^{(1)} & e_4 e_2^{2k-1-j}\mathrm{E}(x_0^{(2)})u_j^{(2)} \end{bmatrix}.$$

V. Correction of the model (1) in the unstable external environment

It is assumed that the estimates of the returned or non-conforming inventory and of unforeseen changes in the deliveries of the inventory used in the model (10) are true, the planned standard behavior of the analyzed production system behavior must be designed for k_0 $(k_0 > 0)$ days, and the admissible threshold ε $(\varepsilon > 0)$ for deviation of the analyzed production system behavior in the unstable external environment from its planed standard behavior is given.Since the random variable z_k $(k \in \mathbf{Z}_+)$ characterizes deviation of the analyzed production system behavior in the unstable environment from its planed standard behavior in the day k , then to ensure the consistency of the model (1) with the model (10) the following step-by-step adaptive approach can be used.

Applying the Monte Carlo Simulation [16] to the models (1) and (10), some set of sequences $z_1^{(j)},\ldots,z_{k_0}^{(j)}$ $(j = 1,\ldots,l)$ can be generated.If $\left\| z_k^{(j)} \right\| \le \varepsilon$ for all $k = 1,\ldots,k_0$ and $j = 1,\ldots,l$, then the model (1) is consistent with the model (10). Therefore, no correction of the model (1) is needed.

Suppose that there exists some integer k_1 $(1 \le k_1 < k_0)$ such that $\left\| z_k^{(j)} \right\| \le \varepsilon$ for all $k = 1,\ldots,k_1$ and $j = 1,\ldots,l$, but $\left\| z_{k_1+1}^{(j)} \right\| > \varepsilon$ for some $j \in \{1,\ldots,l\}$. Then the model (1) is consistent with the model (10) within the first k_1 days.But the model (1) must be corrected starting from the day $k_1 + 1$.

The easiest way to correct the model (1) is to adjust the subsequence $u_{k_1+1},\ldots,u_{k_0}$ so, that the inequality $\left\| z_k^{(j)} \right\| \le \varepsilon$ becomes true for all $j = 1,\ldots,l$ and $k = k_1 + 1,\ldots,k_2$, where k_2 is some maximal integer such that

$k_1 < k_2 \le k_0$. If these actions do not lead to the desired result, then the subsequences of the matrices $A_{k_1+1},\dots,A_{k_0}$ and $B_{k_1+1},\dots,B_{k_0}$ can be adjusted so, that the inequality $\|z_k^{(j)}\| \le \varepsilon$ becomes true for all $j=1,\dots,l$ and $k=k_1+1,\dots,k_2$, where k_2 is some maximal integer such that $k_1 < k_2 \le k_0$.

Remark 5.The sequence of matrices G_k $(k \in \mathbf{Z}_+)$ cannot be corrected when planning the behavior of the analyzed production system, since this sequence is determined by the external environment. For this reason the same sequence of matrices G_k $(k \in \mathbf{Z}_+)$ is used in the models (1) and (10).

If $k_2 < k_0$, the described above procedure is performed for the integer k_2+1 and so on, until the inequality $\|z_k^{(j)}\| \le \varepsilon$ becomes true for all $k=1,\dots,k_0$ and $j=1,\dots,l$.

When using the proposed procedure for correcting the model (1), at least the following two problems arise.

The first problem relates to the complexity associated with the use of the entire set of sequences $z_1^{(j)},\dots,z_{k_0}^{(j)}$ $(j=1,\dots,l)$ obtained via the Monte Carlo Simulation, since this set can be large enough.This complexity can be reduced as follows.Applying bounded reachability analysis [17, 18], we can determine some foreseeable subset of the most significant sequences in the entire set of sequences $z_1^{(j)},\dots,z_{k_0}^{(j)}$ $(j=1,\dots,l)$. Thereafter, the above proposed procedure is applied to this subset of sequences.

The second problem relates to minimizing the costs associated with adjusting the subsequences of ordered delivery and the subsequences of matrices.This is a complex optimization problem, since any adjusting the subsequences of the matrices A_k and B_k results in changing the structure of the analyzed production system.The exact solution to this problem can be obtained using the Branch-and-Bound Method.If it is infeasible to compute the exact solution for problem of minimizing the costs associated with adjusting the subsequences of ordered delivery and the subsequences of matrices, then the Branch-and-Bound Method can be applied for computing some approximate solution of this problem.

VI. Conclusion

The main aim of the given paper was to define and investigate some analytical linear discrete-time stochastic model intended for simulation the behavior of the analyzed production system in the unstable external environment.An essential feature of this model is that it explicitly presentsdestabilizing effects on both the inventory and the external supplies of the analyzed production system.Theoretical analysis of the proposed model was carried out by investigation its average and the second order behaviors, as well as the asymptotic of these behaviors.

One of the most important problems for practice is adaptive correction of the planned standard behavior of the analyzed production system, when this system is functioning in the unstable external environment.To solve this problem, the step-by-step adaptive procedure is proposed.Some methods for reducing complexity when applying this procedure, as well as some methodsof minimization the costs for the analyzed production system related to using the results of this procedure are proposed.Detailed algorithmic analysis and software implementation of the proposed approach is one of the possible areas for future research.

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