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A third order convergent method for solving nonlinear equations

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Abstract
Derivation of the Newton-Raphson method involves an indefinite integral of the derivative of the function, and the relevant area is approximated by a rectangle. In this study, the area under the curve which is appearing in the derivation of Newton-Raphson method is approximated by two points Gaussian quadrature formula. With the help of that an improvement to the Newton-Raphson method is presented for root finding of one variable nonlinear equation. This iterative method converges to the root faster than the Newton-Raphson method and the claim is proved by showing the new method is third order convergent. The Established theory is supported by computed results by applying the new method to a wide range of functions and comparing it with the Newton's method and some third order iterative methods.

Keywords: *Newton's method, Gaussian quadrature formula, Iterative methods, Number of iterations, Order of convergence*

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I. Introduction

Problem of solving nonlinear equations may arise in many situations. The problem is called a root finding problem and it consists of finding values of the variable x that satisfies the equation $f(x) = 0$, for a given function f. A solution to this problem is called a zero of f or a root of $f(x) = 0$. This is one of the oldest numerical approximation problems and yet research is continuing actively in this area even at present. The most popular numerical technique is the Newton-Raphson method, basically developed by Sir Isaac Newton over 300 years ago.

Newton's method, which approximates the roots of a nonlinear equation in one variable using the value of the function and its derivative, in an iterative fashion, is probably the best known and most widely used algorithm and it converges to the root quadratically.

In this study we suggest an improvement to the iterations of Newton's method. Derivation of Newton's method involves an indefinite integral of the derivative of the function and the relevant area is approximated by a rectangle. In the proposed method, we approximate this indefinite integral by applying a Gaussian quadrature formula with two nodes, thereby reducing the error in the approximation substantially. It is shown that the order of convergence of the new method is three, and computed results supports this theory. Even though we have shown that the order of convergence is three in this new method, in several cases, computational order of convergence is even higher.

It is also shown that for quite a number of nonlinear functions the number of iterations required for the new method is significantly less than the that of the Newton's method and some third order iterative methods.

II. Newton's Method

The Newton-Raphson (Newton's) method which is also a fixed-point iterative technique is one of the most powerful and well-known numerical methods for solving a root finding problem $f(x) = 0$. There are at least three common ways of introducing Newton's method. The most common approach is use of graphical techniques. Another way is to derive Newton's method as a simple technique to obtain faster convergence than offered by many other types of functional iterations. The third means of introducing Newton's method, which will be discussed below [3], is an intuitive approach based on Taylor polynomial.

Consider a real interval $[a, b]$ and suppose that $f \in C^2[a, b]$. Let $\overline{x} \in [a, b]$ be an approximation to a root p such that $f'(\overline{x}) \neq 0$ and $|\overline{x} - p|$ is "small". Consider the first-degree Taylor polynomial for $f(x)$, expanded about $\overline{x}$,

$$f(x) = f(\overline{x}) + (x - \overline{x})f'(\overline{x}) + \frac{(x-\overline{x})^2}{2} f''(\zeta(x)), \tag{1}$$

where $\zeta(x)$ lies between x and $\overline{x}$. Since $f(p) = 0$, (1) with $x = p$, gives

$$0 \approx f(\overline{x}) + (p - \overline{x})f'(\overline{x}) + \frac{(p-\overline{x})^2}{2} f''(\zeta(p)). \tag{2}$$

Newton's method is derived by assuming that the term involving $(p-\overline{x})^2$ is negligible and that

$$0 \approx f(\overline{x}) + (p-\overline{x})f'(\overline{x}), \quad (3)$$

Solving for p in this equation gives;

$$p \approx \overline{x} - \frac{f(\overline{x})}{f'(\overline{x})}, \quad (4)$$

which should be a better approximation to p in $\overline{x}$. This sets the stage for the Newton's method, which involves generating the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n = 0,1,2,\cdots \quad (5)$$

The local linear model at x_n is given by

$$M_n(x) = f(x_n) + f'(x_n)(x - x_n) \quad (6)$$

The local model can be interpreted in another way (i.e., by Newton's theorem) as follows

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\theta)\, d\theta, \text{ where } \int_{x_n}^{x} f'(\theta)\, d\theta = f'(x_n)(x - x_n) \quad (7)$$

III. A Third Order Convergence Method (New Method)

From Newton's theorem, we have

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\theta)\, d\theta$$

Consider Gaussian quadrature formula with limits from -1 to 1, then

$$\int_{-1}^{1} f'(x)\, dx \approx \sum_{i=1}^{n} c_i f'(x_i). \quad (8)$$

For a real interval (c, b) taking x with parameter t

$$x(t) = \frac{(d-c)t + d + c}{2} \Longrightarrow dx = \frac{(d-c)}{2} dt, \quad (9)$$

then

$$\int_{c}^{d} f'(x)\, dx = \frac{(d-c)}{2} \int_{-1}^{1} f'\left(\frac{(d-c)t+d+c}{2}\right) dt. \quad (10)$$

For $n = 2$, (8) gives

$$\int_{-1}^{1} f'(x)\, dx = c_1 f'(x_1) + c_2 f'(x_2). \quad (11)$$

Letting $f'(x) = x^3, x^2, x$ and 1, we can obtain $c_1 = c_2 = 1$ and $x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}$.
Then, by (11)

$$\int_{-1}^{1} f'(x)\, dx = f'\left(-\frac{1}{\sqrt{3}}\right) + f'\left(\frac{1}{\sqrt{3}}\right). \quad (12)$$

As a result of assigning $c = x_n$ and $d = x$ and substituting (12) into (10)

$$\int_{x_n}^{x} f'(\theta)\, d\theta = \frac{(x-x_n)}{2}\left[f'\left(-\frac{(x-x_n)}{2\sqrt{3}} + \frac{(x+x_n)}{2}\right) + f'\left(\frac{(x-x_n)}{2\sqrt{3}} + \frac{(x+x_n)}{2}\right)\right] \quad (13)$$

From the definition of Newton's theorem and (13)

$$\begin{aligned} f(x) &= f(x_n) + \int_{x_n}^{x} f'(\theta)\, d\theta \quad (14)\\ &= f(x_n) + \frac{(x-x_n)}{2}\left[f'\left(-\frac{(x-x_n)}{2\sqrt{3}} + \frac{(x+x_n)}{2}\right) + f'\left(\frac{(x-x_n)}{2\sqrt{3}} + \frac{(x+x_n)}{2}\right)\right]\\ &= f(x_n) + \frac{(x-x_n)}{2}\left[f'(ax_n + bx) + f'(bx_n + ax)\right], \end{aligned}$$

where $a = \frac{1}{2} + \frac{1}{2\sqrt{3}}$ and $b = \frac{1}{2} - \frac{1}{2\sqrt{3}}$.
We take the next iterative point as the root $(x = x_{n+1})$ of the local model $f(x_{n+1}) = 0$, then by (14)

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(ax_n + bx_{n+1}) + f'(bx_n + ax_{n+1})} \quad (15)$$

Finally we can define a new scheme which generates the sequence $\{x_n\}$ of iterations for finding a root of $f(x) = 0$ as follows

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(ax_n + bx_{n+1}^*) + f'(bx_n + ax_{n+1}^*)}, n = 0,1,2,\cdots \tag{16}$$

where $a = \frac{1}{2} + \frac{1}{2\sqrt{3}}$ and $b = \frac{1}{2} - \frac{1}{2\sqrt{3}}$ and x_{n+1}^* appears in right hand side is obtained by Newton's methods as follows:

$$x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2,\cdots. \tag{17}$$

IV. Order of Convergence of New Method

Theorem 4.1

Let $f: D \longrightarrow \mathbb{R}$ be a real valued function for an open interval D. Assume that f has first, second and third derivatives in the interval D. If $f(x)$ has a simple root at $\alpha \in D$ and x_0 is sufficiently close to α, then the new method which is given by (16) satisfies the following error equation

$$e_{n+1} = c_2^2 e_n^3 + \mathcal{O}(e_n^4) \tag{18}$$

where $e_n = \alpha - x_n$ and $c_i = (\frac{1}{i!})\frac{f^{(i)}(\alpha)}{f^{(1)}(\alpha)}, i = 1,2,3,\cdots$.

Proof.

The suggested iterative scheme is

$$x_{n+1} = x_n - \frac{2f(x_n)}{f^{(1)}(ax_n + bx_{n+1}^*) + f^{(1)}(bx_n + ax_{n+1}^*)}, n = 0,1,2,\cdots \tag{19}$$

where $a = \frac{1}{2} + \frac{1}{2\sqrt{3}}$ and $b = \frac{1}{2} - \frac{1}{2\sqrt{3}}$ and x_{n+1}^* appears in right hand side is obtained by Newton's methods as follows:

$$x_{n+1}^* = x_n - \frac{f(x_n)}{f^{(1)}(x_n)}, n = 0,1,2,\cdots. \tag{20}$$

In here a and b satisfy $a + b = 1$ and $a^2 + b^2 = \frac{2}{3}$. Let α be a simple root of $f(x)$ (i.e., $f(\alpha) = 0$ and $f^{(1)}(\alpha) \neq 0$ and $x_n = \alpha + e_n$.

Then by using Taylor expansion,

$$\begin{aligned} f(x_n) &= f(\alpha + e_n) \\ &= f(\alpha) + f^{(1)}(\alpha)e_n + \frac{1}{2!}f^{(2)}(\alpha)e_n^2 + \frac{1}{3!}f^{(3)}(\alpha)e_n^3 + \mathcal{O}(e_n^4) \\ &= f^{(1)}(\alpha)(e_n + \frac{1}{2!}\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)}e_n^2 + \frac{1}{3!}\frac{f^{(3)}(\alpha)}{f^{(1)}(\alpha)}e_n^3 + \mathcal{O}(e_n^4)) \\ &= f^{(1)}(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + \mathcal{O}(e_n^4)) \\ &= f^{(1)}(\alpha)(e_n + c_2e_n^2 + \mathcal{O}(e_n^3)). \end{aligned} \tag{21}$$

Further for $f^{(1)}(x_n)$,

$$\begin{aligned} f^{(1)}(x_n) &= f^{(1)}(\alpha + e_n) \\ &= f^{(1)}(\alpha) + f^{(2)}(\alpha)e_n + \frac{1}{2!}f^{(3)}(\alpha)e_n^2 + \mathcal{O}(e_n^3) \\ &= f^{(1)}(\alpha)(1 + \frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)}e_n + \frac{1}{2!}\frac{f^{(3)}(\alpha)}{f^{(1)}(\alpha)}e_n^2 + \mathcal{O}(e_n^3)) \\ &= f^{(1)}(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + \mathcal{O}(e_n^3)). \end{aligned} \tag{22}$$

Let substitute the results of (21) and (22) into (17)

$$\begin{aligned} x_{n+1}^* &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= (\alpha + e_n) - (e_n + c_2e_n^2 + \mathcal{O}(e_n^3))(1 + 2c_2e_n + 3c_3e_n^2 + \mathcal{O}(e_n^3))^{-1} \\ &= \\ &(\alpha + e_n) - (e_n + c_2e_n^2 + \mathcal{O}(e_n^3)) \times [1 - (2c_2e_n + 3c_3e_n^2 + \mathcal{O}(e_n^3)) + \\ &\quad (2c_2e_n + 3c_3e_n^2 + \mathcal{O}(e_n^3))^2 - \cdots] \\ &= (\alpha + e_n) - (e_n + c_2e_n^2 + \mathcal{O}(e_n^3))(1 - 2c_2e_n + \mathcal{O}(e_n^2)) \\ &= (\alpha + e_n) - (e_n - c_2e_n^2 + \mathcal{O}(e_n^3)) \end{aligned} \tag{23}$$

$$= \alpha + c_2 e_n^2 + \mathcal{O}(e_n^3).$$

From (18) and (23)

$$\begin{aligned} ax_n + bx_{n+1}^* &= a(\alpha + e_n) + b(\alpha + c_2 e_n^2 + \mathcal{O}(e_n^3)) \\ &= (a+b)\alpha + ae_n + bc_2 e_n^2 + \mathcal{O}(e_n^3) \\ &= \alpha + ae_n + bc_2 e_n^2 + \mathcal{O}(e_n^3). \end{aligned} \tag{24}$$

Again by taking Taylor expansion of (24)

$$\begin{aligned} &f^{(1)}(ax_n + bx_{n+1}^*) = \\ &f^{(1)}(\alpha) + (ae_n + bc_2 e_n^2 + \mathcal{O}(e_n^3))f^{(2)}(\alpha) + \frac{1}{2!}(ae_n + bc_2 e_n^2 + \\ &\mathcal{O}(en3))2f(3)(\alpha)+\mathcal{O}(en3) \\ &= f^{(1)}(\alpha)[1 + 2(ae_n + bc_2 e_n^2 + \mathcal{O}(e_n^3))c_2 + 3(a^2 e_n^2 + \mathcal{O}(e_n^3))c_3 + \mathcal{O}(e_n^3)] \\ &= f^{(1)}(\alpha)[1 + 2ac_2 e_n + (2bc_2^2 + 3a^2 c_3)e_n^2 + \mathcal{O}(e_n^3)]. \end{aligned} \tag{25}$$

Similarly from (18) and (23)

$$\begin{aligned} bx_n + ax_{n+1}^* &= b(\alpha + e_n) + a(\alpha + c_2 e_n^2 + \mathcal{O}(e_n^3)) \\ &= (a+b)\alpha + be_n + ac_2 e_n^2 + \mathcal{O}(e_n^3) \\ &= \alpha + be_n + ac_2 e_n^2 + \mathcal{O}(e_n^3). \end{aligned} \tag{26}$$

Again by taking Taylor expansion of (26)

$$\begin{aligned} f^{(1)}(bx_n + ax_{n+1}^*) &= f^{(1)}(\alpha) + (be_n + ac_2 e_n^2 + \mathcal{O}(e_n^3))f^{(2)}(\alpha) + \\ &\qquad \frac{1}{2!}\left(be_n + ac_2 e_n^2 + \mathcal{O}(e_n^3)\right)^2 f^{(3)}(\alpha) + \mathcal{O}(e_n^3) \\ &= f^{(1)}(\alpha)[1 + 2(be_n + ac_2 e_n^2 + \mathcal{O}(e_n^3))c_2 + 3(b^2 e_n^2 + \mathcal{O}(e_n^3))c_3 + \mathcal{O}(e_n^3)] \\ &= f^{(1)}(\alpha)[1 + 2bc_2 e_n + (2ac_2^2 + 3b^2 c_3)e_n^2 + \mathcal{O}(e_n^3)]. \end{aligned} \tag{27}$$

From the results of (25) and (27)

$$\begin{aligned} &f^{(1)}(ax_n + bx_{n+1}^*) + f^{(1)}(bx_n + ax_{n+1}^*) = \\ &f^{(1)}(\alpha)[1 + 2ac_2 e_n + (2bc_2^2 + 3a^2 c_3)e_n^2 + \mathcal{O}(e_n^3)] + f^{(1)}(\alpha)[1 + 2bc_2 e_n + (2ac_2^2 + \\ &3b^2 c_3)e_n^2 + \mathcal{O}(e_n^3)] \\ &= f^{(1)}(\alpha)[2 + 2(a+b)c_2 e_n + (2(a+b)c_2^2 + 3(a^2+b^2)c_3)e_n^2 + \mathcal{O}(e_n^3)] \\ &= 2f^{(1)}(\alpha)[1 + c_2 e_n + (c_2^2 + c_3)e_n^2 + \mathcal{O}(e_n^3)]. \end{aligned} \tag{28}$$

From (21) and (28)

$$\begin{aligned} &\frac{2f(x_n)}{f^{(1)}(ax_n + bx_{n+1}^*) + f^{(1)}(bx_n + ax_{n+1}^*)} = \frac{e_n + c_2 e_n^2 + +c_3 e_n^3 + \mathcal{O}(e_n^4)}{1 + c_2 e_n + (c_2^2 + c_3)e_n^2 + \mathcal{O}(e_n^3)} \\ &= [e_n + c_2 e_n^2 + +c_3 e_n^3 + \mathcal{O}(e_n^4)][1 + c_2 e_n + (c_2^2 + c_3)e_n^2 + \mathcal{O}(e_n^3)]^{-1} \\ &= [e_n + c_2 e_n^2 + +c_3 e_n^3 + \mathcal{O}(e_n^4)][1 - (c_2 e_n + (c_2^2 + c_3)e_n^2 + \mathcal{O}(e_n^3)) + (c_2 e_n + \mathcal{O}(e_n^2))^2 - \cdots] \\ &= [e_n + c_2 e_n^2 + +c_3 e_n^3 + \mathcal{O}(e_n^4)][1 - c_2 e_n - (c_2^2 + c_3)e_n^2 + c_2^2 e_n^2 + \mathcal{O}(e_n^3)] \\ &= [e_n + c_2 e_n^2 + +c_3 e_n^3 + \mathcal{O}(e_n^4)][1 - c_2 e_n - c_3 e_n^2 + \mathcal{O}(e_n^3)] \\ &= e_n + c_2 e_n^2 + c_3 e_n^3 - c_2 e_n^2 - c_2^2 e_n^3 - c_3 e_n^3 + \mathcal{O}(e_n^4) \\ &= e_n - c_2^2 e_n^3 + \mathcal{O}(e_n^4). \end{aligned} \tag{29}$$

Finally substituting (29) into proposed scheme (16)

$$x_{n+1} = x_n - \frac{2f(x_n)}{f^{(1)}(ax_n + bx_{n+1}^*) + f^{(1)}(bx_n + ax_{n+1}^*)} \tag{30}$$

$$e_{n+1} + \alpha = e_n + \alpha - (e_n - c_2^2 e_n^3 + \mathcal{O}(e_n^4)) \tag{31}$$

$$\boldsymbol{e_{n+1} = c_2^2 e_n^3 + \mathcal{O}(e_n^4)} \tag{32}$$

Equation (32) establishes the **third order convergence** of the proposed numerical scheme for finding roots of univariate nonlinear equation.

□

V. Computational Results

In order to approximate roots of various functions such as polynomial, exponential, trigonometric, and combinations of those, we employed Newton-Raphson method (NRM), Multi-point iterative method 1 (MIM 1), Multi-point iterative method 2 (MIM 2), and Chebyshev method (CHM) in [4] together with this new method (see Table 1). The efficiency of these methods were compared by the number of iterations required to reach the approximate root to a same number of decimal places starting from same initial guess x_0. Note that frequently the new method (NM) converges to the root faster than NRM. Compared to the third order methods (MIM 1, MIM 2, CHM), we notice that in most cases the new method takes same number of iterations or a lesser number of iterations.

Even though we have shown that (see Table 2) the order of convergence is three, in quite number of cases, computational order of convergence is even higher. Note that some values are not defined (ND), not converged (NC), not converge to the same root (NR) under some iterative schemes.

Table 1. Computed results: Number of iterations

Function	x_0	Root	Number of Iterations				
			NRM	NM	MM1	MM2	CM
$x^3 + 4x^2 - 10$	-0.3	1.3652300134	53	4	17	49	8
$(x-1)^3 - 1$	-1.2	2	18	8	6	6	22
$x^3 - 10$	-1.9	2.1544346900	12	4	4	16	21
$x^3 + x^2 - x - 1$	-1.6	1.83928867552	33	6	7	11	26
$-x^3 - 1.3x^2 + 0.4x - 0.03$	0.4	-1.5674093215	131	7	43	47	76
$x^5 - 1.5x^3 + 23x^2 - x + 12$	13	-3.0796963106	55	9	19	42	15
$cosx - x$	7	0.7390851332	50	5	18	3	5
$e^x - 2^{-x}ln2 - 2sinx$	-0.65	-1.65560004485	36	4	16	12	6
$-x^3 + 2x^2 + 10x - 20$	-17	-3.1622776602	8	5	5	6	6
$x + 3sinx - 2$	10	4.9295434476	82	4	NR	125	NC

Table 2. Computed results: Order of convergence

Function	x_0	Root	Number of Iterations				
			NRM	NM	MM1	MM2	CM
$x^3 + 4x^2 - 10$	-0.3	1.3652300134	2	2.84	2.73	2.81	2.76
$(x-1)^3 - 1$	-1.2	2	1.95	3.15	3.18	2.77	2.52
$x^3 - 10$	-1.9	2.1544346900	1.97	3.04	2.91	1.55	2.71
$x^3 + x^2 - x - 1$	-1.6	1.83928867552	1.98	2.96	2.77	2.75	3.42
$-x^3 - 1.3x^2 + 0.4x - 0.03$	0.4	-1.5674093215	1.97	2.54	2.84	2.95	2.72
$x^5 - 1.5x^3 + 23x^2 - x + 12$	13	-3.0796963106	2.00	2.92	2.91	2.69	2.60
$cosx - x$	7	0.7390851332	1.99	3.47	3.46	ND	3.08
$e^x - 2^{-x}ln2 - 2sinx$	-0.65	-1.65560004485	1.97	2.94	3.43	2.87	2.49
$-x^3 + 2x^2 + 10x - 20$	-17	-3.1622776602	1.98	2.54	2.58	2.83	2.88
$x + 3sinx - 2$	10	4.9295434476	1.96	2.99	NR	2.61	NC

VI. Conclusion

Newton's method is probably the most well-known and widely used algorithm for solving nonlinear equations. It converges to the root quadratically. By carefully investigating the behaviour and derivation of Newton's method, we suggest an improvement for Newton's method. Derivation of Newton's method involves an indefinite integral of the derivative of the function and the relevant area is approximated by a rectangle. In our study, this rectangle area is approximated by Gaussian quadrature formula with two nodes. By doing so, we have derived a new method for function of one variable and established results are shown in (16). Analysis of convergence in the section 4 shows that the new iterative method has third order convergence, that is, the sequence x_n generated by this method converges to a root with the order of convergence three. Computational order of convergence of the new method was almost three for most of the functions, we tested. For considerable number of functions computational order of convergence was even more than three.

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