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On some double sequence spaces and matrix class characterizations

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Abstract
The study is aimed at introducing a new double sequence space, $\overline{sl_2(p)}$ *and transform its sequences into other sequence spaces. The methods of introducing this new sequence space is mainly by Moricz extension techniques which involves constructions of some theorems which helped us establish some topological properties of* $\overline{sl_2(p)}$*. After introducing the double sequence space,* $\overline{sl_2(p)}$*, the study has proved that it is a paranormed double sequence space. The study is also able to determine its* $\alpha-, \beta-$ *and* $\gamma-$ *duals, and characterized several four dimensional matrix classes involving the new double sequence space.*
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I. Introduction

Let any sequence (x_n) in a sequence space, X be converted to a sequence (y_n) belonging to another sequence space, Y by means of the transformation

$$y_n = \sum_{k=1}^{\infty} a_{n,k} x_k \tag{1.1}$$

where $A = (a_{n,k})$ is an infinite matrix and y_n exists as $n \to \infty$. Then we can write $A \in (X, Y)$, where (X, Y) is called matrix class, see (Cooke, 1950, p. 58 and Maddox, 1970, p. 161 - 162). The ordinary sequences (x_n) and (y_n) were extended to double sequences $(x_{k,l})$ and $(y_{k,l})$, the matrix $A = (a_{n,k})$ was also extended to four the dimensional matrix $A = (a_{m,n,k,l})$ as in (Robison, 1926 and Hamilton, 1936) and the transformation (1.1) becomes

$$y_{mn} = \sum_{k,l=1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \tag{1.2}$$

These extensions resulted in generating double sequence spaces and characterizations of four dimensional matrices on double sequence spaces. The theory of double sequences was motivated by the fact that convergent double sequences *need not* be bounded in the sense of Pringsheim; see (Pringsheim, 1900, Hardy 1903 and Hardy 1916). This study is motivated by the work of Moricz (Moricz, 1991) and because of the strong statement: "*The crucial difference between the convergence of ordinary sequences and the convergence in Pringsheim's sense is that the later does not imply boundedness of double sequences in question*", (Pringsheim, 1900). So, extensions and generalizations of single to double sequences initiated by Pringsheim and later by Hardy and other researchers are well defined works. Thus, this paper wishes to extend the ordinary sequence spaces $\overline{Sl(p)}$ defined by Mishra (Mishra, et al, 2015) to double sequence spaces, $\overline{Sl_2(p)}$. Furthermore, we extend the ordinary sequence space Γ to the double sequence space Γ_2, Γ^* to the double sequence Γ_2^* and σ_∞ to the double sequence σ_∞^2, respectively. Some properties and duals of these double sequences would be developed. Four dimensional matrix classes involving these double sequence spaces will also be characterized.

Some Notations in literatures:

Let w^2 denote the set of all complex double sequences. The Maddox-Simmons ordinary sequence spaces $l(p)$ were extended to the double sequence spaces $l_2(p)$ in the paper (Gokhan and Colak, 2006). In fact, let $p = (p_{mn})$ be a double sequence of strictly positive real numbers p_{mn}, then $l_2(p) = \{x = (x_{mn}) \in w^2 : \sum_{m,n}^{\infty,\infty} |x_{mn}|^{p_{mn}} < \infty\}$ and is proved to be a linear space if $sup_{m,n>1} p_{mn} < \infty$. Further, let $H = sup_{m,n>1} p_{mn} < \infty$ and $M = \max\{1, H\}$. Then double sequence spaces $l_2(p)$ was paranormed by $g(x) = \left(\sum_{m,n}^{\infty,\infty} |x_{mn}|^{p_{mn}}\}\right)^{\frac{1}{M}}$ and proved to complete linear metric space. Γ is the space of sequences $x = \{x_p\}$ such that $|x|^{\frac{1}{p}} \to 0$, as $p \to \infty$. The space Γ can be regarded as the space of all integral functions $f(z) = \sum_{p=1}^{\infty} x_p z^p$, Γ^* *is* the space of sequences $s = \{ s_p \}$ such that the sequence $\{|s_p|^{\frac{1}{p}}\}$ is bounded, and σ_∞ is any perfect sequence space of matrices defined as $\sigma_\infty = \{A = (a_{nk}): |a_{nk}| < M\}$. The spaces $l(p), \overline{l(p)}, \overline{sl(p)},$ and $l_2(p)$ are of great interest in this study and are clearly defined as follows: let an infinite matrix $A = (a_{nk})$ be given by $a_{nk} = \begin{cases} 2^{-n}, & n = k \\ 0, & \text{otherwise} \end{cases}$. Then the sequence space $\overline{Sl(p)}$ was defined to be: $\overline{sl(p)} = \{x = (x_k): \sum_{k=1}^{\infty} \frac{1}{2^k} |t_k(x)|^{p_k} < \infty\}$. So, $\overline{sl(p)} = \{x = (x_k): Ax \in \overline{l(p)}\}$. Thus, $\overline{sl(p)}$ is now the set of all sequences $\{v_k\}$ whose A- transforms are in the sequence space $\overline{l(p)}$ with $\overline{sl(p)}$= $\overline{[l(p)]}_A$. Here the sequence $\{v_k\}$ is given by $\{v_k\}$ =$\sum_{r=1}^{k} \frac{1}{2^r} |t_r(x)|^{pr}$ $\overline{sl(p)}$. It was proved to be isomorphic to $\overline{l(p)}$ and also seen to be a complete paranormed space, paranormed by $\left(\sum_{k=1}^{\infty} \frac{1}{2^k} |t_k(x)|^{p_k}\right)^{\frac{1}{M}}$, where $M = \max\{\frac{1}{2}, sup_k \frac{p_k}{2^k}\}$.

The $\alpha-$, $\beta - and$ $\gamma -$ duals of double sequence spaces and the concept of summability domains can be found in (Basar and Colak, 2012, p. 295 - 296). They are defined as follows:

$X^\alpha = \{(a_{ij}) \in \omega^2 : \sum_{i,j} |a_{i,j} x_{i,j}| < \infty \text{ for all } (x_{i,j}) \in X \}$,

$X^\beta = \lambda^{\beta(v)} = \{(a_{ij}) \in \omega^2 : \sum_{i,j} a_{i,j} x_{i,j} \text{exists for all } (x_{i,j}) \in X \}$,

$\lambda^\gamma = \{(a_{i,j}) \in w^2 : sup_{k,l} |\sum_{i,j}^{k,l} a_{i,j} x_{i,j}| < \infty \; for \; all \; (x_{ij}) \in \lambda\}$

For any two sequence spaces λ and μ of double sequences, $Y^\eta \subset X^\eta$ whenever $X \subset Y$, where $\eta \in \{\alpha, \beta\}$. And v-summability domain $X_A^{(v)}$ of four dimensional infinite matrix in a space X is defined by

$X_A^{(v)} = \{x = (x_{kl}) \in \omega^2 : Ax(v - \sum_{k,l} a_{mnkl} x_{kl})$ for $m, n, k, l \in \mathbb{N}$ and is in $X\}$

II. Results:

We define the double sequence $\overline{sl_2(p)}$ as follows:

$\overline{sl_2(p)} = \{x = (x_{kl}) \in \omega^2 : \sum_{k,l}^{\infty,\infty} 2^{-kl} |t_{kl}(x)|^{p_{kl}} < \infty\}$ 4.1

σ_∞^2 can be defined as a double sequence space of four dimensional matrices defined by:

$$\sigma_\infty^2 = \{A \equiv (a_{mn,kl}): |a_{mnkl}| < M\}$$

Following (Hardy, 1916), a double sequence $x = (x_{mn})$ is said to converge regularly if it converges in Pringsheim`s sense and, in addition the following finite limits exist:

$$\lim_{n\to\infty} x_{mn} = K_m \quad (m = 1,2, \dots) \tag{4.2}$$

$$\lim_{m\to\infty} x_{mn} = L_n \quad (n = 1,2, \dots) \tag{4.3}$$

Γ_2 is the space of double sequence $x = (x_{mn})$ such that $|x_{mn}|^{1/n} \to 0$ as $m, n \to \infty$. The space Γ_2 can be regarded as the space of all integral functions,

$$f(z) = \sum_{m,n=1}^{\infty,\infty} x_{mn} z^n \tag{4.4}$$

Γ_2^* is the space of double sequence $s = (s_{m,n})$ such that the double sequence $\{|s_{mn}|^{1/n}\}$ is bounded. Γ_2^* may also be considered as the space conjugate to Γ_2 which is regarded as the space of integral functions. Each continuous linear functional $U \in \Gamma_2^*$ is of the form

$$U(f) = \sum_{m,n=1}^{\infty,\infty} s_{mn} z_n \tag{4.5}$$

Next, the two theorems that follow immediately describe the linear and topological properties of the double sequence space in (4.1).

Theorem 1: The space $\overline{sl_2(p)}$ is linearly isomorphic to the double sequence space $\overline{l_2(p)} = x = (x_{kl}) \in \omega^2 : \sum_{k,l}^{\infty,\infty} |t_{kl}(x)|^{p_{kl}} < \infty\}$.

Proof: Now, for each $\in \overline{sl_2(p)}$, $Sx \in \overline{l_2(p)}$, where $S = (s_{mnkl})$ is a four dimensional matrix. Moreover, S is linear and bijective. Also the matrix $T = (t_{mnkl}) = S^{-1}$. Which proves that $\overline{sl_2(p)}$ is linearly isomorphic to $\overline{l_2(p)}$.

Theorem 2: The space $\overline{sl_2(p)}$ is complete paranormed space paranormed by $g(t) = [\sum_{k,l}^{\infty,\infty} 2^{-kl} |t_{kl}(x)|^{p_{kl}}]^{1/M}$, where $M = \max(1, H)$ with $H = sup_{k,l\geq 1} p_{kl} < \infty$.

Proof: First we prove that $\overline{sl_2(p)}$ is complete. For this let $(t^{ij}(x))$ be a Cauchy sequence in $\overline{sl_2(p)}$, where$t^{ij}(x) = (t_{kl}^{ij}(x))_{k,l \in N}$. Then for every $\in > 0$, there exists a positive integer $N = N(\in)$ such that

$$g\left(t^{ij}(x) - t^{uv}(x)\right) = \left[\sum_{k,l}^{\infty,\infty} 2^{-kl} \left|t_{kl}^{ij}(x) - t_{kl}^{uv}(x)\right|^{p_{kl}}\right] < \varepsilon \tag{4.6}$$

for all $i, j, u, v > N$. From this we can infer that

$$2^{-kl} \left|t_{kl}^{ij}(x) - t_{kl}^{uv}(x)\right|^{p_{kl}} \leq g\left(t_{kl}^{ij}(x) - t_{kl}^{uv}(x)\right) < \varepsilon$$

for every fixed $k, l \in N$ and for every positive integers $k, lu, v > N$. Therefore, $t_{kl}^{ij}(x), i, j \in N$ is a Cauchy sequence in $\mathbb{C}$ is convergent to $t_{kl}(x)$, say. So that

$\lim_{i,j\to\infty} t_{kl}^{ij}(x) = t_{kl}(x)$ (4.7)

Getting $t_{kl}(x)$, we define $t(x) = t_{kl}(x)$, and we may write

$$\sum_{k,l}^{P,Q} 2^{-kl}|t_{kl}(x)|^{p_{kl}} < \mathcal{E}^M, (P, Q = 1, 2, 3, \dots)$$

for $k, l, u, v > N$ by (4.6). Letting $u, v \to \infty$, and then $P, Q \to \infty$ we obtain

$\sum_{k,l}^{\infty,\infty} 2^{-kl}\left|t_{kl}^{ij}(x) - t(x)\right|^{p_{kl}} < \mathcal{E}^M$ for $i, j > N$ by (4.7) Hence $g\left(t^{ij}(x) - t(x)\right) < \mathcal{E}$

for all $i, j > N$. Which proves completeness.

Next, we prove that $\overline{sl_2(p)}$ is a paranormed space paranormed by

$g(t) = [\sum_{k,l}^{\infty,\infty} 2^{-kl}|t_{kl}(x)|^{p_{kl}}]^{1/M}$ (4.8)

where $M = \max(1, H)$ with $H = sup_{k,l\geq 1} p_{kl} < \infty$. Now, it is trivial that $g(\theta) = 0$, $g(-t) = g(t)$, where $t = t_{kl}(x)$. And for any $t, t' \in \overline{sl_2(p)}$, we write

$g(t + t') \leq [\sum_{k,l}^{\infty,\infty} |t_{kl}(x)|^{p_{kl}}]^{1/M} + [\sum_{k,l}^{\infty,\infty} |t'_{kl}(x)|^{p_{kl}}]^{1/M} = g(t) + g(t')$ (4.9)

Next, clearly for any complex number λ, we have $g(\lambda t) \to t$ is continuous at $\lambda = 0$, $t = \theta$ and that if λ is fixed the function $t \to \lambda t$ is continuous. This proves the theorem.

Theorem 3. The α-dual of the spaces Γ_2 and Γ_2^* is the space $\overline{sl_2(p)}$.

Proof: Let $\lambda \in \{\Gamma_2, \Gamma_2^*\}$. since $\lambda \subset \sigma_\infty^2$ and $(\sigma_\infty^2)^\alpha = \overline{sl_2(p)}$, the validity of the inclusion

$\overline{sl_2(p)} \subset \lambda^\alpha$ (4.11)

is clear. Conversely , suppose that $a = (a_{i,j}) \notin \overline{sl_2(p)}$. Then one can easily see in the special case $x = (x_{i,j}) = \{(-1)^{i+j}\} \in \Gamma_2$ that $\sum_{ij}|a_{ij}x_{ij}| = \sum_{ij}|a_{ij}| = \infty$.

This means that $a \notin {\Gamma_2}^\alpha$ which contradicts the hypothesis. Hence, $a = (a_{i,j})$ must be in $\overline{sl(p)}^2$ that is to say that the inclusion

$\Gamma_2^\alpha \subset \overline{sl_2(p)}$ (4.12)

must be hold. By combining the inclusions (4.11) and (4.12) the desired result immediately follows .

In similar way, the inclusion $\Gamma_2^{*\alpha} \subset \overline{sl_2(p)}$ and $\Gamma_2 \subset \overline{sl_2(p)}$ easily hold .This completes the proof .

Now we may give the $\beta - duals$ of the spaces of double series with respect to the $v -$ convergence for $v \in \{p, bp, r\}$ using the technique of (Basar, 2003), and (Basar and Altay, 2002), for the single sequence.

Lemma 7. The matrix $A = (a_{mnkl})$ is $c_v -$ conservative for $v = r$ iff

$sup_{m,n\in N} \sum_{k,l}|a_{mnkl}| < \infty$, (4.13)

$\vartheta - \lim_{m,n\to\infty} a_{mnkl} = \propto_{k,l}$ $(k, l \in \mathbb{N})$ (4.14)

$\vartheta - \lim\limits_{m,n\to\infty} \sum\limits_k a_{mnkl_0} = ul_0$ $(l_0 \in \mathbb{N})$ and

$\vartheta - \lim_{m,n\to\infty} \sum_l a_{mnk_0l} = vk_0$ $(k_0 \in \mathbb{N})$ (4.15)

$\vartheta - \lim_{m,n\to\infty} \sum_{k,l} a_{mnkl} = v$ (4.16)

Theorem 4. The $\beta(r) -$ dual of the space $\overline{sl_2(p)}$ is the set σ_∞^2 .

Proof: Suppose that $x = (x_{mn}) \in \overline{sl_2(p)}$. Then the double sequence $S = (s_{mn})$, defined by

$s_{mn} = \sum_{i,j=0}^{mn} x_{ij}$ $(m, n \in \mathbb{N})$, is in the space $l(p)$.

Let us determine the necessary and sufficient condition in order for the series

$\sum_{i,j} a_{i,j}x_{i,j}$ (4.17)

to be $r -$ convergent for a sequence $a = (a_{i,j}) \in \omega^2$. We obtain by applying the Abel generalized transformation for double sequences to the m, nth partial sums of the series in (4.17) that

$z_{m,n} = \sum_{i,j}^{m,n} a_{i,j}x_{i,j}$

$= \sum_{i,j=0}^{m-1,n-1} s_{i,j}\Delta_{11}a_{i,j} + \sum_{i=0}^{m-1} s_{i,n}\Delta_{10}a_{i,n} + \sum_{j=0}^{n-1} s_{m,j}\Delta_{01}a_{m,j} + s_{m,n}a_{m,n}$ (4.18)

for all $m, n \epsilon \mathbb{N}$ (4.18) may be rewritten by the matrix representation

$z_{m,n} = \sum_{i,j}^{m,n} b_{mnij}s_{i,j} = (BS)_{mn})$ (4.19)

for all $m, n \epsilon \mathbb{N}$, where the four $-$ dimensional matrix $B = (b_{mnij})$ is defined by

$$b_{mnij} = \begin{cases} \Delta_{11}a_{i,j}\,, & 0 < i \leq m-1 \ and \ 0 < j \leq n-1, \\ \Delta_{10}a_{i,n}, & 0 < i \leq m-1 \ and \quad j = n\,, \\ \Delta_{01}a_{m,j}\,, & i = m \ and \ 0 < j\,, \\ a_{mn}\,, & i=m \quad and \ j=n\,, \\ 0, & \text{otherwise} \end{cases} \qquad (4.20)$$

We therefore read from the equality (4.20) that $ax = (a_{mn}x_{mn}) \in \overline{sl_2(p)_r}$ $whenever\ x = (x_{mn}) \in \overline{sl_2(p)_r}$ iff $z = (z_{mn}) \in \overline{l_2(p)_r}$ $whenever\ \ s = (s_{mn}) \in \overline{l_2(p)_r}$ which leads us to the fact that $B = (b_{mnij})$ defined by (4.20) is in the class $\overline{(l_2(p)_r}\ :\ \overline{l_2(p)_r}\,)$ .Thus we see from lemma (4.9) for the matrix B, defined above, that the conditions

$sup_{m,n\in\mathbb{N}} \sum_{k,l=0}^{m,n} |b_{mnkl}| = sup_{m,n\in\mathbb{N}} \{\sum_{k,l=0}^{m-1,n-1} |\Delta_{11}a_{k,l}| + \sum_{k=o}^{m-1} |\Delta_{10}a_{kn}| + \sum_{l=0}^{n-1} |\Delta_{01}a_{ml}| + |a_{mn}|\} < \infty$,

$$r - \lim_{m,n\to\infty} b_{mnkl} = \Delta_{11}a_{k,l}(k, l \in \mathbb{N}),$$

$$r - \lim_{m,n\to\infty} \sum_{k} b_{mnkl_0} = \Delta_{01}a_{1l_0}(l_0 \in \mathbb{N}) ,$$

$$r - \lim_{m,n\to\infty} \sum_{l} b_{mnk_0l} = \Delta_{10}a_{k_01}(k_0 \in \mathbb{N}), and$$

$$r - \lim_{m,n\to\infty} \sum_{k,l} b_{mnkl} = a_{00} \qquad hold.$$

Therefore, we derive from the conditions (4.13) – (4.16) that

$\sum_{k,l} |\Delta_{11}a_{k,l}| < \infty,$ (4.21)

$sup_{n\in\mathbb{N}} \sum_{k} |\Delta_{10}a_{kn}| < \infty\ and$ (4.22)

$sup_{m\in\mathbb{N}} \sum_{l} |\Delta_{01}a_{ml}| < \infty$ (4.23)

Further, since the equality

$\sum_{k=0}^{m} |b_{mnk0}| = \sum_{k=0}^{m-1} |\Delta_{10}a_{k0}| + |a_{m0}|$ (4.24)

hold, the supremum over $m \in \mathbb{N}$ of this expression gives that $(a_{k0})\ k \in \mathbb{N} \in b_v$ and similarly for m = 0 equality

$\sum_{l=0}^{n} |b_{mn0l}| = \sum_{l=0}^{n-1} |\Delta_{01}a_{0l}| + |a_{0n}|$ (4.25)

holds which also gives by taking supremum over $n \in \mathbb{N}\ that\ \ (a_{0l})\ \ l \in \mathbb{N} \in b_v$,

where b_v denotes the spaceof single sequences of bounded variation.

Beside by using the equality

$\Delta_{10}a_{kn} = \Delta_{10}a_{k0} - \sum_{l=0}^{n-1} \Delta_{11}a_{k,l}$, $\qquad we\ see\ that$

$\sum_{k} |\Delta_{10}a_{kn}| = \sum_{k} |\Delta_{10}a_{k0} - \sum_{l=0}^{n-1} \Delta_{11}a_{k,l}| \le \sum_{k} |\Delta_{10}a_{k0}| + \sum_{l=0}^{n-1} |\Delta_{11}a_{k,l}|$ (4.26)

Since it is seen by (4.24) that $\sum_{k} |\Delta_{10}a_{k0}| < \infty$ and (4.24) implies (4.22) when (4.21) considered in the equality (4.26).

Similarly by employing the equality $\Delta_{01}a_{ml} = \Delta_{01}a_{0l} - \sum_{k=0}^{m-1} \Delta_{11}a_{k,l}$, one can see from the equality

$\sum_{l} |\Delta_{01}a_{ml}| = \sum_{l} |\Delta_{01}a_{0l} - \sum_{k=0}^{m-1} \Delta_{11}a_{k,l}| \le \sum_{l} |\Delta_{01}a_{0l}| + \sum_{l} |\sum_{k=0}^{m-1} \Delta_{11}a_{k,l}|$

through (4.21) and (4.25) that (4.23) holds.

There it is deduced that the conditions on the sequence $a = (a_{ij}) \in \Omega$ in order for the series in (4.17) to be $r - convergent$ are $\sum_{k,l} |\Delta_{11}a_{k,l}| < \infty$, $(a_{k0})\ \ k \in \mathbb{N} \in b_v$.

This shows that $\overline{sl(p)}_r^{\beta(r)} = \sigma_\infty^2$, which completes the proof.

Theorem 5: The $\beta(\vartheta) - dual$ of the space Γ_2 is the set $\{a = (a_{i,j}) \in \omega^2\ ; \mathrm{B} = (b_{mnij}) \in (\Gamma_2^\eta : \Gamma_2^\vartheta)\}$, where $\mathrm{B} = (b_{mnij})$ is defined by (4.12).

As the consequences of theorem (4.12), $\Gamma_{2_r}^{\beta(p)} = \Gamma_{2_r}^{\beta(r)} = Bv$,

$\Gamma_{2_{bp}}^{\beta(bp)} = \{(a_{k,l}) \in \omega^2 : \sum_{k,l} |\Delta_{11}a_{k,l}| < \infty\ , (\Delta_{10}a_{i,n})n \in \mathbb{N}, (\Delta_{01}a_{m,j})\ m \in \mathbb{N} \in c_0(i, j \in \mathbb{N})\}$,

and, $\Gamma_{2_{bp}}^{\beta(bp)} = \{(a_{k,l}) \in \omega^2 : \sum_{k,l} |\Delta_{11}a_{k,l}| <$ *∞,, Δ10ak,l l∈N∈c0k∈N,*

where c_0 denotes the space of ordinary null sequences.

Theorem 5: In order that $\{y_{m,n}\}$ should belong to Γ_2^* whenever $\{x_{k,l}\}$ belong to Γ_2, it is necessary and sufficient the double sequence $\{x_{m,n}\}$ is bounded where $x_{m,n}$, $(m, n = 1,2, \ldots)$. Where $\mathcal{A} \in (\Gamma_2, \Gamma_2^*)$.

Proof: If

$|a_{mnkl}|^{1/mn} \to 0$ as $m, n \to \infty$ uniformly in k, l (4.31)

For sufficiency: Since $\{x_{k,l}\} \in \Gamma_2$, there exist a finite $k \ge 1$ such that

$\sum_{k,l=1}^{\infty,\infty} |x_{k,l}| \le k$ (4.32)

By (4.31) given $\mathcal{E} > 0$, we can find $N = N(\mathcal{E})$ independent of k, l such that

$|a_{mnkl}|^{1/mn} < \frac{\mathcal{E}}{2k}$ for $m, n > N$ and all k, l (4.33)

By (4.32) and (4.33) we have

$|y_{mn}|^{\frac{1}{mn}} = \left|\sum_{k,l=1}^{\infty,\infty} a_{mnkl}x_{kl}\right|^{\frac{1}{mn}} \le \left(\sum_{k,l=1}^{\infty,\infty}|a_{mnkl}||x_{kl}|\right)^{\frac{1}{mn}} \le \left(\frac{\mathcal{E}}{2k}\right)K^{\frac{1}{mn}}$ (since $k \ge 1$ $and, l = 1$)

$$\le \left(\frac{\mathcal{E}}{2k}\right)k = \frac{\mathcal{E}}{2} < E \quad , for\ n > N$$

(Necessity) .Suppose that (4.31) is not satisfied.

Then for some $\mathcal{E} > 0$ $\ there\ exist\ no\ N\ such\ that |a_{mnkl}|^{\frac{1}{mn}} \ge \mathcal{E}\ \ for\ n > N\ and\ k, l = 1,2, \ldots$

That is for this $\mathcal{E}$ and any N there is an $n > N\ \ and\ \ k, l\ such\ that\ |a_{mnkl}|^{\frac{1}{mn}} \ge \mathcal{E}$.

Theorem 6: Let $0 < p_{kl} \le \frac{1}{2}$ for every $k, l \in M, N$. Then $\mathcal{A} \in \left(\overline{sl_2(p)},\ \sigma_\infty^2\right)$ if and only if .

i. $sup_n \left|\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{mnkl}\left(-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_v}} + \right.\right.$ $N-22N1pk<\infty\ ;\ N\ge 1$.

ii. $sup_m \left|\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{mnkl}\left(-\sum_{u=1}^{l-1}\left(\frac{m^{-2}}{2^m}\right)^{\frac{1}{p_u}} + \right.\right.$ $m-22m1pl<\infty\ ;\ M\ge 1$.

iii. $sup_{n,k}|\Delta a_{mnkl}|^{p_k} < \infty\ where\ \ \Delta a_{mnkl} = a_{mnkl} - a_{mnk+1,l}$.

iv. $sup_{m,k}|\Delta a_{mnkl}|^{p_l} < \infty\ where\ \ \Delta a_{mnkl} = a_{mnkl} - a_{mnk,l+1}$.

Proof: Let the condition hold. Now

$$\left|\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{mnkl}\left(-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_v}} + \left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_v}}\right.\right|$$

$$\le sup_n \left|\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{mnkl}\left(-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_v}} + \left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_k}}\right)\right| < \infty\,.$$

i.e.

$\sum_{k=1}^{\infty} a_{mnkl}\left(-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_v}} + \left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_k}}\right)$converges.

It implies that $\mathcal{A}_{mn} \in \overline{sl(p)^2}$ and hence $|\mathcal{A}_{mn}(x)| = \left|\sum_{k,l=1}^{\infty} a_{mnkl}\, x_{kl}\right| < \infty$. Conversely, let $\mathcal{A} \in \left(\overline{sl_2(p)},\ \sigma_\infty^2\right)$. Since,

$\delta = \left\{-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_v}} + \left(\frac{N^{-2}}{2^N}\right)\right\} \in \ \overline{sl_2(p)}\,, we\ have\ \mathcal{A}_{mn}\,\sigma < \infty\,,$ for each$n \in N\ ,\ m \in M\ and\ \ that\ \mathcal{A}_{mn}\,\sigma < \sigma_\infty^2$.

So,

$sup_n\left|\sum_{k=1}^{\infty} a_{mnkl}\sigma\right| < \infty$. And$sup_m\left|\sum_{l=1}^{\infty} a_{mnkl}\sigma\right| < \infty$.

Let us define the matrix $B = (b_{mnkl})\ \ by\ \ b_{mnkl} = \Delta\, a_{mnkl}\ ;\quad n, k \in N\ , m, l \in M$.

Then $B \notin \left(\overline{sl_2(p)}\,, \sigma_\infty^2\right)$ by the fact that when $0 < p_{kl} \le 1\ for\ every\ k \in N\ , l \in M$, then $\mathcal{A} \in \left(\overline{sl_2(p)}\ ,\ \sigma_\infty^2\right)\ if\ and\ only\ if\ sup_m|a_{mnkl}|^{p_{kl}} < \infty$. Hence there is a sequence $y = (y_{kl}) \in \overline{sl_2(p)}$ such that $\sum_{k=l=1}^{\infty} b_{mnkl}\, y_{kl} \ne 0$. However if we define the sequence $U = (U_{kl})\quad by$

$$U_{kl} = \begin{cases} y_{\frac{1}{2},l} & for\ k = 1, \\ y_{k,\frac{1}{2}} & for\ \ l = 1, \\ \frac{y_{kl}}{2^{kl}} - \frac{y_{k-1,l}}{2^{k-1,l}} & for\ k > 1, \\ \frac{y_{kl}}{2^{kl}} - \frac{y_{k,l-1}}{2^{k,l-1}} & for\ l > 1\,. \end{cases}$$

Then $U \in \ \overline{sl_2(p)}$ and $\sum_{k=l=1}^{\infty} a_{mnkl}\, U_{kl} = \sum_{k=l=1}^{\infty} b_{mnkl}\ \ y_{kl}/2^{kl} \ne 0$, which is now a contradiction to the fact that $\mathcal{A} \in \left(\overline{sl_2(p)}\,, \sigma_\infty^2\right)$. Hence we must have, $sup_{m,l}|\Delta a_{mnkl}|^{p_l} < \infty,\ \ sup_{n,k}|\Delta a_{mnkl}|^{p_k} < \infty$.

Theorem 7: Let $0 < p_{kl} \leq \frac{1}{2}$ $for\ every\ k \in N, l \in M. Then\ \mathcal{A} \in \left(\overline{sl_2(p)}\,, \Gamma_2\right)$ $if\ and\ only\ if$

i. $\mathcal{A}_{mn}\left(-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_v}} + \left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_k}}\right) \in \Gamma_2, N > 1$;

ii. $\mathcal{A}_{mn}\left(-\sum_{u=1}^{k-1}\left(\frac{M^{-2}}{2^M}\right)^{\frac{1}{p_u}} + \left(\frac{M^{-2}}{2^M}\right)^{\frac{1}{p_l}}\right) \in \Gamma_2,\ M > 1;$

iii. $B \in \left(\overline{sl(p)^2}\,, \Gamma_2\right)$ $where\ B = (b_{mnkl}) = (\Delta a_{mnkl});\ \ m, n \in M\,, k, l \in N.$

iv. $\lim_{n\to\infty} \Delta a_{mnkl} = \Delta\propto_{kl}\ (k\ is\ fixed).$

v. $\lim_{m\to\infty} \Delta a_{mnkl} = \Delta\propto_{kl}\ (l\ is\ fixed).$

Proof:

Let us assume that the above conditions hold.Then for any $x = (x_{kl}) \in \overline{sl_2(p)}$.
Then for any $\sum_{k=l=1}^{\infty} a_{mnkl}x_{kl}$ is absolutely convergent, and that ,
$\sum_{k=l=1}^{\infty} a_{mnkl} = \sum_{k=l=1}^{\infty} \propto_{kl} x_{kl}$. Hence $\mathcal{A} \in \left(\overline{sl_2(p)}\,, \Gamma_2\right)$.
Conversely, let $\mathcal{A} \in \left(\overline{sl_2(p)}\,, \Gamma_2\right)$. Then $\sum_{k=l=1}^{\infty} a_{mnkl}x_{kl}$ converges for each $x = (x_{kl}) \in \overline{sl_2(p)}$ and for $n, m \in N$. If we define the double sequence. $V = (V_{kl})\ by$

$$V_{kl} = \begin{cases} z_{\frac{1}{2},l} & for\ k = 1, \\ z_{k,\frac{1}{2}} & for\ \ l = 1, \\ \frac{z_{kl}}{2^{kl}} - \frac{z_{k-1,l}}{2^{k-1,l}} & for\ k > 1, \\ \frac{z_{kl}}{2^{kl}} - \frac{z_{k,l-1}}{2^{k,l-1}} & for\ l > 1\,. \end{cases}$$

Then it can easily be verified that $V \in \overline{sl_2(p)}$ and $\Delta a_{mnkl} \to \Delta\propto_{kl}\ (as\ n \to \infty) U \in \overline{sl_2(p)}$ and $\Delta a_{mnkl} - \Delta\propto_{kl}\ \ (as\ m \to \infty)$ since $x = \left(-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_v}} + \left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_k}}\right) \in \overline{sl_2(p)}$. Then the necessity follows; we need to show that $B \in \left(\overline{sl_2(p)}\,, \Gamma^2\right)$. On the contrary we assume that $B \notin \left(\overline{sl_2(p)}\,, \Gamma^2\right)$. It can easily be verified that;
$\left(\sum_{k=l=1}^{\infty} a_{mnkl}u_{kl}\right) = \left(\sum_{k=l=1}^{\infty} b_{mnkl}y_{kl}\right) \notin \Gamma^2$where $y = (y_{kl}) \in \overline{sl_2(p)} and\ u = (u_{kl}) \in \overline{sl_2(p)}$.
This is a contradiction to the fact that $B \in \left(\overline{sl_2(p)}\,, \Gamma_2\right)$. This proves the necessity.

Theorem 8: $\mathcal{A} \in \left(\overline{sl_2(p)}, \Gamma_2^*\right)$ if and only if $sup_{m,n}\left(\sum_{kl}\left|\frac{a_{mnkl}}{v_{kl}}\right| K^{\frac{s}{p_{kl}}}(MN)^{-\frac{1}{p_{kl}}}\right)^{q_{mn}} < \infty$
for some $M > 1, N > 1$.

Proof :

Sufficiency.

Let $x = (x_{kl}) \in \overline{sl_2(p)}$. Then there exist $k_1$and l_1 such that $|v_{kl}x_{kl}| < K^{\frac{s}{p_{kl}}}(MN)^{-1/p_{kl}}$ for some $M > 1\,, N > 1\ and\ k > k_1\,, l > l_1$.
Hence for every m, n we have;
$|\mathcal{A}_{mn}x|^{q_{mn}} \leq l\left|\sum_{k,l=1}^{k_1\,l_1} a_{mnkl}x_{kl}\right|^{q_{mn}} + l\left|\sum_{k>k_1,l>l_1} a_{mnkl}x_{kl}\right|^{q_{mn}} = l(S_1 + S_2),$ where
$l = \max(1, 2^{H-1})\,, H = sup_{mn}q_{mn}S_1 = \left(\left|\sum_{k,l=1}^{k_1\,l_1} a_{mnkl}x_{kl}\right|\right)^{q_{mn}} = \left(\left|\sum_{k,l=1}^{k_1\,l_1}(a_{mnkl}/v_{kl})v_{kl}x_{kl}\right|\right)^{q_{mn}}$

$$\leq \left(\sum_{k>k_1,l>l_1}\left|\frac{a_{mnkl}}{v_{kl}}\right| K^{\frac{s}{p_{kl}}}(MN)^{-\frac{1}{p_{kl}}} \max_{k>k_1,l>l_1}|v_{kl}x_{kl}|(MN)^{p_{kl}} K^{\frac{-s}{p_{kl}}}\right)^{q_{mn}} < \infty$$

$$\leq \left(\sum_{k>k_1,l>l_1}\left|\frac{a_{mnkl}}{v_{kl}}\right| L^{\frac{s}{p_{kl}}}(MN)^{-\frac{1}{p_{kl}}} \max_{k>k_1,l>l_1}|v_{kl}x_{kl}|(MN)^{p_{kl}} L^{\frac{-s}{p_{kl}}}\right)^{q_{mn}} < \infty.$$

For the sum $S_2\,, we\ have$;

$$S_2^{\ 1/q_{mn}} = \left|\sum_{k>k_1,l>l_1} a_{mnkl}x_{kl}\right| = \left|\sum_{k>k_1,l>l_1}(a_{mnkl}/v_{kl})v_{kl}x_{kl}\right|$$
$$\leq \sum_{k>k_1,l>l_1}\left|\frac{a_{mnkl}}{v_{kl}}\right|(KL)^{\frac{s}{p_{kl}}}N^{-\frac{1}{p_{kl}}}\ .$$

Hence $S_2\ \leq T\,,\ thus\ \mathcal{A}_{mn}x \in \Gamma_2^*\ and\ \mathcal{A} \in \left(\overline{sl_2(p)}, \Gamma_2^*\right)$.

Theorem 9: $\mathcal{A} \in (\Gamma_2, \sigma_\infty^2)$, and $\sigma_\infty^2 = \sigma_r^2$ is the set of matrices $A = (a_{mnkl})$ such that $\sum_{m,n=1}^{\infty\infty} \left|\sum_{k,l\in E} a_{mnkl}\right|^r \leq M^r$., where E is an arbitrary subset of the positive integers and M is independent of E.

Proof: To prove necessity, let E be an arbitrary subset of the positive integers. Let Γ_2 be the set of all $x = \{x_{kl}\}$ such that $|x_{kl}| \leq 1$ $(k, l = 1,2 \ldots)$. Then Γ_2 *is* $P-$bounded in σ_∞^2 Hence if $A \equiv (a_{mnkl}) \in (\Gamma_2, \sigma_\infty^2)$, then AX is $P-$bounded in σ_∞^2, i.e, if $y = \mathcal{A}x$ for $x \in \Gamma^2$, then $\sum_{mn}|y_{mn}|^r \leq M^r$, where M is a constant for every r.

Thus, $\sum_{m,n=1}^{\infty\infty} |\sum_{k=l=1} a_{mnkl}x_{kl}|^r \leq M^r$ *for every* $x \in \Gamma_2$. Take

$x_{kl} = 1$, $k, l \in E$ *and* $x_{kl} = 0$ otherwise then $x \in \Gamma_2$ and hence,

$\sum_{m,n=1}^{\infty\infty} \left|\sum_{k,l\in E} a_{mnkl}x_{kl}\right|^r \leq M^r$, and the condition has been proved necessary.

Theorem 10: $\mathcal{A} \in (\Gamma_2^*, \sigma_\infty^2)$ if and only if $\left(\sum_{k,l}\left|\frac{a_{mnkl}}{v_{kl}}\right| K^{\frac{s}{p_{kl}}} N^{\frac{1}{p_{kl}}}\right)^{q_{mn}} \to 0$ as $m, n \to \infty$ for every integer $M > 1$, $N > 1$.

Proof : Let $x \in \Gamma_2^*$. So that $sup_{kl} k^{-s}|v_{kl}x_{kl}|^{p_{kl}} < \infty$ and $sup_{kl} l^{-s}|v_{kl}x_{kl}|^{p_{kl}} < \infty$.

Choose $M > \max(1, sup_{kl} k^{-s}|v_{kl}x_{kl}|^{p_{kl}})$, $N > \max(1, sup_{kl} l^{-s}|v_{kl}x_{kl}|^{p_{kl}})$. Then

$$|\mathcal{A}_{mn}(x)|^q n \leq \left(\sum_{k,l}\left|\frac{a_{mnkl}}{v_{kl}}\right| |v_{kl}x_{kl}|\right)^{q_{mn}}$$

$$\leq \left(\sum_{k,l}\left|\frac{a_{mnkl}}{v_{kl}}\right| K^{\frac{s}{p_{kl}}} (MN)^{\frac{1}{p_{kl}}}\right)^{q_{mn}} \to 0 \text{ as } m, n \to \infty$$

Hence, $\mathcal{A}_{mn}(x) \in \sigma_\infty^2$ and $\mathcal{A} \in (\Gamma_2^*, \sigma_\infty^2)$.

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