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Oscillatory Solutions for Some Quasilinear Second Order Elliptic One-Dimensional Equations: Some Estimates and Comparison Results

TADIE

To T. Karsten, T-K Christoffer and C.E Kanstrup

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Abstract. This work is about investigating the estimates of the diameters and amplitudes of the nodal sets of of the strongly oscillatory (and decaying) solutions for the quasilinear differential one-dimensional equations of the type

$$\left(a(t)|u'|^{\alpha-1}\ u'\right)' + c(t)|u|^{\beta-1}u + b(t)|u'|^{\alpha-1}\ u' + f(t,u,u') = 0 \quad \text{in } \mathbb{R}^+$$

where the coefficients a and c are strictly positive and continuous functions ; $\beta,\ \alpha > 0$ and "admissible" function $f \in C(\mathbb{R}^3)$. These equations are the one-dimensional versions the multidimensional ones

$$\nabla\cdot\left\{A(x)\left(|\nabla w|^{\alpha-1}\ \nabla w\right)\right\} + C(x)\left[|w|^{\beta-1}\ w\right] + B(x)\cdot\left(|\nabla w|^{\alpha-1}\ \nabla w\right) + F(x,w,\nabla w) = 0, \quad x \in \mathbb{R}^n.$$

This work shows that as concerned those estimates for such decaying solutions (e.g. when $\lim_{t\to\infty} \dfrac{c(t)}{a(t)} = 0$; see [3, 6]) the admissible perturbations elements (f or F) do not interfere.

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1. Introduction

1.1. Preliminaries. .

In the sequel we define the following notations and operators:

$\forall\alpha > 0,\quad s \in \mathbb{R}$ and $S \in \mathbb{R}^n \quad \phi_\gamma(s) := |s|^{\alpha-1}s$ and $\Phi_\alpha(S) := |S|^{\alpha-1}S$ and after easy and elaborate calculations, they have the forrlowing properties:

$$\begin{cases} s\phi_\alpha(s) = |s|^{\gamma+1}; \quad s\phi'_\alpha(s) = \alpha\phi_\alpha(s) \quad and \quad \phi_\alpha(st) = \phi_\alpha(s)\phi_\alpha(t); \\ S\Phi_\alpha(S) = |S|^{\alpha+1} \quad and \quad \Phi_\alpha(TS) = \Phi_\alpha(T)\Phi_\alpha(S). \end{cases} \tag{1.1}$$

Also for any $\gamma,\ \beta > 0$ and the regular functions $u \in C^2(\mathbb{R})$ and $v \in C^2(\mathbb{R}^n),\quad n \geq 2,$ we define the following operators :

$$
\begin{aligned}
p_\gamma(u) &:= \left\{a(t)\phi_\gamma(u')\right\}' + c(t)\phi_\gamma(u) \quad ; \; t > 0 \quad \text{and} \\
Q_\beta(v) &:= \nabla \cdot \left\{A(x)\Phi_\beta(\nabla v)\right\} + C(x)\phi_\beta(v) \quad ; \; x \in \mathbb{R}^n; \\
\pi_{\gamma,\beta}(u) &:= \left\{a(t)\phi_\gamma(u')\right\}' + c(t)\phi_\beta(u) \quad ; \; t > 0 \quad \text{and} \\
Q_{\gamma,\beta}(v) &:= \nabla \cdot \left\{A(x)\Phi_\gamma(\nabla v)\right\} + C(x)\phi_\beta(v) \quad x \in \mathbb{R}^n;
\end{aligned}
\tag{1.2}
$$

the coefficients being continuous and positive functions. The operators with only one index (like p_α) are called halflinear operators and they are homogenous in the sense that $P_\alpha(ku) = k^\alpha P_\alpha(u) \quad \forall k \in \mathbb{R}$. This work will concern only one-dimensional cases.

For a continuous function w, any bounded and close interval $D := D(w)$ will be called a nodal set of w if

$$
w(x) \neq 0 \quad \text{inside} \quad D(w) \quad \text{and} \quad w|_{\partial D} = 0 \; . \tag{1.3}
$$

Obviously if w is replaced by w^+ , $w(x) \neq 0$ will be replaced by $w(x) > 0$ (and by $w(x) < 0$ for the $D(w^-)$ case) in (1.3).

In the sequel the general hypotheses are:

(H)

H1) The coefficients of the principal part a and corresponding c are assumed to be continuous, positive and bounded away from zero.

With $t \in \mathbb{R}^+$, u defined on $\mathbb{R}$, $x \in \mathbb{R}^n$, and w defined on $\mathbb{R}^n$

H2) For the perturbations terms like $f(t, u, u')$ and $F(x, w, \nabla w)$,

i) $f \in C(\mathbb{R}, \mathbb{R}, \mathbb{R})$ is positive or $\forall s \in \mathbb{R}$, $sf(t, s, u') \geq 0$;

ii) $F(x, w, \nabla w)$ is positive or $\forall w \in \mathbb{R} \quad wF(x, w, \nabla w) \geq 0$.

H3) If like in $(\Pi_{\gamma,\beta})$ the operator has bouble indices, the index carried by the second order part, γ here is the leading index. In that case the perturbation term, f or F say, must satisfy

$$
\lim_{\xi \to \infty} \frac{f(t, \xi, u')}{\phi_\gamma(\xi)} = 0.
$$

1.2. **First results related to the operators above via some Picone-type formulas.** Consider for $i = 1, 2$ the equations

$$
\begin{cases}
P_i(y) := \left\{a_i(t)\phi(y_i')\right\}' + c_i(t)\phi(y_i) + f_i(t, y_i, y_i') = 0, \quad t \in \mathbb{R} \quad \text{and} \\
Q_i(u_i) := \quad \nabla \cdot \left\{A_i(x)\Phi(\nabla u_i)\right\} + C_i(x)\phi(u_i) + F_i(x, u_i, \nabla u_i) = 0, \quad x \in \mathbb{R}^n.
\end{cases}
$$

Let y_1 and y_2 be respectively solutions of $P_i(y_i),\ i=1,2$. Then wherever y_2 is non zero, a version of Picone's identity reads

$$
\textbf{(P)}\quad \begin{cases} \left\{ y_1 a_1(t)\phi(y_1') - y_1\phi(\dfrac{y_1}{y_2})a_2(t)\phi(y_2') \right\}' = \\ = a_2(t)\zeta_\alpha(y_1,y_2) + \Big[a_1(t)-a_2(t)\Big]|y_1'|^{\alpha+1} + \Big[q_2(t)-q_1(t)\Big]|y_1|^{\alpha+1} \\ + |y_1|^{\alpha+1}\left[\dfrac{f_2(t,y_2,y_2')}{\phi(y_2)} - \dfrac{f_1(t,y_1,y_1')}{\phi(y_1)}\right] \end{cases}
$$

where, $\forall\gamma>0$, the two-form function ζ_γ is defined $\forall u,v\in C^1(\mathbb{R},\ \mathbb{R})$ by

$$
(\mathbf{Z1}):\qquad \zeta_\gamma(u,v)\begin{cases} = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\dfrac{u}{v}v') + \gamma v'\dfrac{u}{v}\phi_\gamma(\dfrac{u}{v}v') \\ = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\dfrac{u}{v}v') + \gamma|\dfrac{u}{v}v'|^{\gamma+1} \end{cases}
$$

is strictly positive for non null $u\neq v$ and null only if $u=\lambda v$ for some $\lambda\in\mathbb{R}$. Similarly, if u_1 and u_2 are respectively solutions of $Qu_i,\ i=1,2$, then wherever u_2 is non zero, a version of Picone's identity reads

$$
\textbf{(Q)}\quad \begin{cases} \nabla\cdot\left\{ u_1A_1(x)\Phi(\nabla u_1) - u_1\phi(\dfrac{u_1}{u_2})A_2(r)\Phi(\nabla u_2)\right\} = A_2(r)Z_\alpha(u_1,u_2) \\ + \Big(A_1(x)-A_2(r)\Big)|\nabla u_1|^{\alpha+1} + \Big(C_2(r)-C_1(x)\Big)|u_1|^{\alpha+1} \\ + |u_1|^{\alpha+1}\left[\dfrac{F_2(x,u_2,\nabla u_2)}{\phi(u_2)} - \dfrac{F_1(x,u_1,\nabla u_1)}{\phi(u_1)}\right] \\ \text{where}\quad \forall\gamma>0,\quad \forall u,v\in C^1(\mathbb{R}^n) \\ \textbf{(Z2)}:\quad Z_\gamma(u,v) := |\nabla u|^{\gamma+1} - (\gamma+1)\Phi_\gamma(\dfrac{u}{v}\nabla v)\cdot\nabla u + \gamma|\dfrac{u}{v}\nabla v|^{\gamma+1} \\ = |\nabla u|^{\gamma+1} - (\gamma+1)|\dfrac{u}{v}\nabla v|^{\gamma-1}\,\dfrac{u}{v}\nabla v\cdot\nabla u + \gamma|\dfrac{u}{v}\nabla v|^{\gamma+1}. \end{cases}
$$

We recall that $\forall\gamma>0$ the two-form $Z_\gamma(u,v)\geq 0$ and is null only if $\exists k\in\mathbb{R};\ u=kv$. (see e.g. [1], [2]).

To set a use of Picoine-type formula, we establish the next lemma where the hypotheses are a bit weaker than in earlier results mainly not requiring for the coefficient a to be differentiable.

Lemma A.

For some $\alpha,\ m>0$, *let* $a,\ c\in C\Big(\mathbb{R}^+,\ [\ m,\ +\infty\ )\Big)$. *Then any non-trivial regular solution of*

$$(i)\quad \Big(a(t)\phi_\alpha(u')\Big)' + c(t)\phi_\alpha(u) + F(t,u,u') = 0;\quad t>0$$

(ii) *where* $F\in C(\mathbb{R}^3,\ \mathbb{R}^+)$ *or* $\forall w,\ s\in\mathbb{R}\quad wF(t,w,s)\geq 0$ *and satisfies*

$$\forall w,\ s\in\mathbb{R},\quad \lim_{u\to 0}\frac{F(s,u,w)}{\phi_\alpha(u)} = 0$$

is strongly oscillatory .

Proof. Let A and C be two positive constants (to be determined a posteriori). If we assume that $u > 0$ in some $[R, \infty)$ then the Picone formula from the equations

$$\Big(a(t)\phi_\alpha(u')\Big)' + c(t)\phi_\alpha(u) + F(t,u,u') = 0 \text{ and}$$

$$\Big(A\phi_\alpha(y')\Big)' + C\phi_\alpha(y) = 0 \text{ formally gives}$$

$$\Big\{y\phi_\alpha(y') - y\phi_\alpha(\,\frac{y}{u}u'\,)\,\Big\}' = (\,A - a(t)\,)|y'|^{\alpha+1} + a(t)Z_\alpha(y,u) + |y|^{\alpha+1}\Big(c(t) - C + \frac{F(t,u,u')}{\phi_\alpha(u)}\Big)$$

and the proof is completed by takind in any $[R,\ 10R)$ $A = \max_{(R,\ 10R)}\ a(t)$ and

$C = \min_{(R,\ 10R)}\ c(t)$ leading to the existence of zeros of u inside any such a nodal set

$D(y^+) \subset (R,\ 10R)$.

In fact if we integrate over such a nodal set $D = D(y^+)$, say we get

$$\int_D \Big\{y\phi_\alpha(y') - y\phi_\alpha(\,\frac{y}{u}u'\,)\,\Big\}' dt$$

$$= \int_D (\,A - a(t)\,)|y'|^{\alpha+1} + a(t)Z_\alpha(y,u) + |y|^{\alpha+1}\Big(c(t) - C + \frac{F(t,u,u')}{\phi_\alpha(u)}\Big)dt.$$

This is absurd as since $y|_{\partial D} = 0$, the left hand side is zero while the right is strictly positive. Therefore the assumption is false anb u has a zero in any $(R,\ 10R)$ for any large R.

□

For $\phi := \phi_\alpha$ and $\psi := \phi_\beta$, consider in $\mathbb{R}^+$ the regular solutions u and v of

$$\begin{cases} \Big(a(t)\phi(u')\Big)' + c(t)\phi(u) = 0 \quad \text{and} \\ \Big(a(t)\phi(v')\Big)' + c_1(t)\psi(v) = 0, \quad t > 0 \\ \text{where } a, c \text{ and } c_1 \text{ are strictly positive functions.} \end{cases} \tag{1.4}$$

The version of Picone's formula we use here for the equations in (1.4) reads for functions u and w

$$\begin{aligned} &\Big\{ua(t)\phi(u') - a(t)u\phi(\,\frac{u}{w})\,\phi(w')\Big\}' \\ &= \ a(t)Z_\alpha(u,w) + |u|^{\alpha+1}\Big[c_1(t)|w|^{\beta-\alpha} \ - \ c(t)\Big] \end{aligned} \tag{1.5}$$

where $\forall u, v \in C^1(\mathbb{R}),\ \forall\gamma > 0$ the two-form

$$Z_\gamma(u,v) := |u'|^{\gamma+1} - (\gamma+1)u'\phi(\,\frac{u}{v}v'\,) + \gamma|\frac{u}{v}v'|^{\gamma+1}$$

is non-negative and is zero only if $u \equiv \mu v$ for some $\mu \in \mathbb{R}$.

Definition 1.1. *A function $f \in C(\mathbb{R}^+)$ is said to be oscillatory if it has zeros in any*
$I_R := (R, \infty)$ and strongly oscillatory if has infinitely many nodal sets inside any I_R.

We can now establish the first result. This result had been established in ([4], Theorem 1.3) and the proof here is different and much simple.

Theorem 1.2. *Let two positive and continuous functions a, c be given. Then $\forall\, \beta,\ \alpha > 0$, any non-trivial and bounded solution v of*

$$\pi_{\alpha,\beta}(v) := \Big\{a(t)\phi_\alpha(v')\Big\}' + c(t)\phi_\beta(v) = 0, \quad t > 0 \tag{1.6}$$

is strongly oscillatory in $\mathbb{R}^+$. This remains true even when any continuous and non-negative perturbation term $g(t,u,u')$ satisfying $\dfrac{|g(t,v,v')|}{|\phi_\alpha(v)|} = o(|v|)$ for small $|v|$ is added to the equation.

Moreover the perturbation term would also apply if it is a restoration function in u (i.e. $\forall\, u$, $u\, g(t,u,u') \geq 0$) and $\dfrac{|g(t,v,v')|}{|\phi_\alpha(v)|} = o(|v|)$ for small $|v|$.

Proof. Given the hypotheses on the coefficients a and c, any such a solution of

$$\Big(a(t)\phi(u')\Big)' + c(t)\phi(u) = 0,\ t > 0$$

is strongly oscillatory (see e.g. [4]). Let v be a non-trivial and bounded solution of (1.6).

Assume that $v > 0$ in some I_R; $R > 0$ and let

$$\min_{[R\, < t <\, 10R]} [c(t)v(t)]^{\beta-\alpha} := c_R > 0.$$

Keeping in mind that 10 can be replaced by a bigger number if necessary , the equation

$$\Big(a(t)\phi(u')\Big)' + c_R\phi(u) = 0, \quad t > 0 \tag{1.7}$$

has a multitude nodal sets $D(u^+)$ inside $(R, \quad 10R)$. We choose a nodal set $D_0 := D(u^+) \subset (R, \quad 10R)$. The solution v then satisfies $c(t)[v(t)]^{\beta-\alpha} > c_R$ in D_0. From (1.5) the Picone-type formula for (1.6) and (1.7) reads in D_0 (as $v \neq 0$ there)

$$\begin{aligned} &\Big\{ua(t)\phi(u') - a(t)u\phi(\ \frac{u}{v})\ \phi(v')\Big\}' \\ &= \ a(t)Z_\alpha(u,v) + |u|^{\alpha+1}\Big[c(t)|v|^{\beta-\alpha}\ -\ c_R\Big] \end{aligned} \tag{1.8}$$

where $\quad c(t)|v|^{\beta-\alpha} = c(t)v(t)^{\beta-\alpha} \geq\ c_R.$

The integration of both sides of (18) over D_0 provides a contradiction as the left hand side gives zero and the right a strictly positive value. Therefore v has to have a zero in any $I_R := (R, \infty)$.

When $g(t,v,v')$ is added to the equation (1.6) the Picone formula gives from (1.5) ,the new (1.6) and (1.7)

$$\begin{cases} \Big(a(t)\phi(u')\Big)' + c(t)\phi(u) = 0 \quad \text{and} \\ \Big(a(t)\phi(v')\Big)' + c(t)\psi(v) + g(t,v,v') = 0, \quad t > 0 \\ \Big\{u\phi(u') - u\phi(\frac{u}{v}v')\Big\}' = a(t)Z_\alpha(u,v) + |u|^{\alpha+1}\Big[c(t)|v|^{\beta-\alpha} \ - \ c_R\Big] \\ + |u|^{\alpha+1}\,\dfrac{g(t,v,v')}{\phi(v)}. \end{cases} \tag{1.9}$$

and the conclusion follows. The estimate condition $\dfrac{g(t,v,v')}{|\phi_\alpha(v)|} = o(|v|)$

(i.e. $\lim_{s\to 0} \dfrac{g(t,s,v')}{|\phi_\alpha(s)|} = 0$) is to avoid v from being compactly supported (ie to satisfy $v \equiv 0$ in some I_ρ, $\rho > 0$, (see [10]).

□

The next result is related to problems with damping terms i.e. problems whose perturbation terms contains the group $B(x)\cdot\Phi_\alpha(\nabla u)$ in multidimensional cases or $b_1(t)\phi_\alpha(u')$ for one-dimensional case. Here , for the multidimensional case, $B \in C^1(\mathbb{R}^n\ ,\ \mathbb{R})$ and $b_1 \in C^1(\mathbb{R}\ ,\ \mathbb{R})$ for one-dimensional. It is to be noticed that the damping term is a function of t times the group $\phi_\alpha(v')$ which is present in the principal part of the operator.

We then look at

$$\Big\{a(t)\phi_\alpha(u')\Big\}' + c(t)\phi_\beta(u) + b(t)\phi_\alpha(u') = 0, \quad t > o \tag{1.10}$$

where $b(t)\phi_\alpha(u')$ denotes the damping term. For multidimensional it has the form

$$\nabla\cdot\Big\{A(x)\Phi_\alpha(\nabla U)\Big\} + C(x)\phi_\beta(U) + B(x)\cdot\Phi_\beta(\nabla U) = 0, \quad x \in \mathbb{R}^n.$$

The first result is the

Theorem 1.3. *Let $\beta,\ \alpha > 0$ be given. Assume that as before and $a,\ c \in C(\mathbb{R}^+)$ are strictly positive.*

Then if there is $B \in C^1(\mathbb{R}^+)$ satisfies $B'(t) = b(t)$ the problem (1.10) is strongly oscillatory.

Proof. Let u be a non-trivial and bounded solution for (1.10).

Assume that $\forall T > 0,\ \exists\theta \equiv \theta(T) > 0\ ;\ c(t)|u(t)|^{\beta-\alpha} > \theta \quad \forall t \in [T,\ 20T) := I^{(20R)}$.

For the strongly oscillatory equation

$$\Big\{a(t)\phi_\alpha(y')\Big\}' + \theta\phi_\beta(y) = 0, \quad t > 0$$

we get with (1.10) the following version of Picone-type formula

$$\Big\{ay\phi_\alpha(y') - ay\phi_\alpha(\frac{y}{u}u') - By\phi_\alpha\Big(\frac{y}{u}\Big)\Big\}' \\ = a(t)Z_\alpha(y,u) + |y|^{\alpha+1}\Big[c(t)|u|^{\beta-\alpha} - \theta(R)\Big] - B(t)\Big(y\phi_\alpha(\frac{y}{u})\Big)' \tag{1.11}$$

after some straight forward caculations. For R large enough, there are a multitude of nodal sets $D(y^+) \subset I^{(20R)}$. The integration of (1.11) over any such a $D := D(y^+)$ gives
(as $y|_{\partial D} = 0$)

$$0 = \int_D \Big\{a(t)Z_\alpha(y,u) + |y|^{\alpha+1}\Big[c(t)|u|^{\beta-\alpha} - \theta(R)\Big]\Big\}dt - \int_D B(t)\Big\{y\phi_\alpha\Big(\frac{y}{u}\Big)\Big\}'dt. \tag{1.12}$$

Obviously for any $k \in \mathbb{R}$ (1.10) and (1.11) remain valid if $B(t) + k$ replaces $B(t)$. Thus if $u > 0$ in any $I^{(10)}$, (1.12) holds after such any replacement i.e.

$$\forall k \in \mathbb{R}, \quad \int_D \Big\{a(t)Z_\alpha(y,u) + |y|^{\alpha+1}\Big[c(t)|u|^{\beta-\alpha} - \theta(R)\Big]\Big\}'dt \\ = \int_D [B(t)+k]\Big\{y\phi_\alpha(\frac{y}{u})\Big\}'dt \tag{1.13}$$

which can so hold only if each of the two integrands is null in D. Since the first integrang is strongly positive, it cannot hold. Therefore the assumption is false and u has zeros in any $I^{(20R)}$.

□

If the perturbation $g(t,u,u')$ (where g fulfills the conditions expressed in (1.9)) is added to the equation (1.10) it reads

$$\Big\{a(t)\phi_\alpha(u')\Big\}' + c(t)\phi_\beta(u) + b(t)\phi_\alpha(u') + g(t,u,u') = 0, \quad t > o. \tag{1.14}$$

Corollary 1.4. *If the conditions expressed on g are like those in (1.9), then the conclusions of Theorem 1.3 hold for the problem (1.14).*

Proof. If we suppose that there is a bounded and non-trivial solution u of (1.14) which is strictly positive in any $I^{(10)}$, the proof goes as before as here (1.12) reads

$$0 = \int_D \Big\{a(t)Z_\alpha(y,u) + |y|^{\alpha+1}\Big[c(t)|u|^{\beta-\alpha} - \theta(R) + \frac{g(t,u,u')}{\phi(u)}\Big]\Big\}dt \\ - \int_D B(t)\Big\{y\phi_\alpha\Big(\frac{y}{u}\Big)\Big\}'dt. \tag{1.15}$$

□

2. Estimates for some decaying oscillatory solutions

Remark 2.1. *We will say that two functions (solutions) u and v are* **overlapping** *in D, say if they have two nodal sets $D(u^+)$ and $D(v^+)$ such that $D(u^+)\bigcup D(v^+) \subset D$ and*

$$\exists \xi \in D(u^+)\bigcap D(v^+)\,; \qquad u'(\xi) = v'(\xi) = 0. \tag{2.1}$$

Moreover whenever $D(u^+)\bigcap D(v^+) \neq \emptyset$ there will be such a $\xi \in \mathbb{R}$ such that ether u and V or U and v are overlapping in $D(u^+)\bigcup D(v^+)$ where for any function w, $W(t) := w(t+\xi)$. In the sequel (2.1) will be supposed true whenever overlapping will be concerned.

Consider for some $m > 0$ the equation

$$\begin{cases} \Big(a(t)\phi_\alpha(v')\Big)' + c(t)\phi_\beta(v) + g(t, v, v') = 0,\ t > 0; \\ a,\ c \in C(\mathbb{R}^+,\ (m,\ \infty)\ ); \\ \beta,\ \alpha > 0 \quad \text{and } g \text{ is like in Theorem 1.2.} \end{cases} \tag{2.2}$$

Theorem 2.2. *In addition of the hypotheses on a and c assume that*

$$\text{with } \chi(t) := \frac{a(t)}{c(t)}\,, \qquad \lim_{t\nearrow\infty} \quad \chi(t) = 0. \tag{2.3}$$

Then with $D_T(v^+) := [t_1,\ t_2]$ denoting a nodal set of v^+ inside $I_T := (T,\ \infty)$, as $t \nearrow \infty$,

$$\begin{aligned} &(i) \quad diam\ D(v+) := |t_2 - t_1| = O\Big(\ \chi(t)^{1/(\alpha+1)}\ \Big); \\ &(ii)\ \text{if in addition } a \geq 1 \text{ then also } \max_{t\in D(v+)} v(t) = O\Big(\ \chi(t)^{1/(\alpha+1)}\ \Big). \end{aligned} \tag{2.4}$$

Proof. Let v be a non-trivial and bounded solution of (2.2). v is strongly oscillatory and $\forall k > 0$, so is that of the auxillary equation

$$\Big(a(t)\phi_\alpha(u')\Big)' + kc(t)\phi_\alpha(u) = 0, \quad t > 0. \tag{2.5}$$

As in (1.9) a Picone-type formula from (2.2) and (2.5) reads with $\phi := \phi_\alpha$

$$\begin{aligned} &\Big\{au\phi(u') - au\phi(\ \frac{u}{v}v'\ )\Big\}' = a(t)Z_\alpha(u,v) + |u|^{\alpha+1}\,\frac{g(t,v,v')}{\phi(v)} \\ &+ |u|^{\alpha+1}c(t)\Big[|v(t)|^{\beta-\alpha}\ -\ k\Big]. \end{aligned} \tag{2.6}$$

From the proof of Theorem 1,2, any nodal set $D(u^+)$ overlaps one of $D(v^+)$. Thus for large enough $T > 0$, u^+ and v^+, two solutions from (2.2) and (2.5) overlap i.e.

$\exists \xi \in D(u^+)\bigcap D(v^+)$ such that $u'(\xi) = v'(\xi) = 0$. The right hand side of (2.6) is strictly positive in $D(u^+)$ if we assume that $v > 0$ in $D(u^+)$, using $k := \min_{D(u^+)} |v(t)|^{\beta-\alpha}$.

Let $D_1 := (t_1,\ \xi)$ and $D_2 := (\xi,\ t_2)$ where $D(u^+) := [t_1,\ t_2]$. Thus v^+ has a zero in any of D_i whence $D(v^+) \subset D(u^+)$.

If $\alpha \geq 1$, ϕ_α is monotonic increasing in the sense that
$\forall s, t \in \mathbb{R}, \qquad (t-s)\Big(\phi_\alpha(t) - \phi_\alpha(s)\Big) \geq 0.$

Assume that $J := \{t \in D(u^+) \bigcap D(v^+)\ ;\ v^+ > u^+\}$ has non-zero measure. We chose $\tau > 0$ and $I := I_\tau \subset J$ such that $w(t) := u - v + \tau > 0\ \forall t \in I$ and $w|_{\partial I} = 0$. Then from the equations, for $\phi := \phi_\alpha$

$$\begin{aligned} &\int_I w\Big[a\phi(u') - a\phi(v')\Big]dt \\ &= -\int_I a(t)(u'-v')\Big[\phi(u') - \phi(v')\Big]dt \qquad (2.7) \\ &= \int_I w(\ c\psi(v) + g - kc\phi(u)\ ]dt = \int_I w\Big(c(t)[\ \frac{\psi(v)}{\phi(u)} - k\ ] + \frac{g}{\phi(u)}\Big)dt. \end{aligned}$$

In I_τ, $v^+ > u^+$ so in the last term, $|v|^\beta |u|^{-\alpha} - k \geq |v|^{\beta-\alpha} - k \geq 0$ making the term positive; but the second term , $-\int_I a(t)(u'-v)\Big[\phi(u') - \phi(v')\Big]dt$ is negative. Therefore the assumption cannot hold.

Thus $v^+ \leq u^+$ and $\max_{D(u^+)}\{u^+ - v^+\} \geq 0$ in addition of $D(v^+) \subset D(u^+)$ we got earlier. The estimates for u^+ follow from Theorem 4.1 of [3].

3. Concluding remarks

CR1) In earlier works we did similar investigations concerning the operators like in $\Pi_{\alpha,\beta}$ (without general perturbation terms and with constant coefficient a) (see [3]).

CR2) The use of the Picone formulas even in cases with damped terms led to more general equations (see e.g. [7]).

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