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On The Convergence of Generalized Beta-Kantorovich Type Operators

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Abstract: *In the presented article, we generalize the positive linear operator which is Kantorovich type with beta bases given by Dhawal. J. Bhatt et al. recently. We introduce the real positive parameters to generalize the operator, then prove uniform convergence for the sequence of this newly defined operators with the help of Korovkin's theorem. We estimate central and pointwise moments and also the rate of convergence through modulus of continuity. We have shown its asymptotic behaviors in terms of Voronovskaya type asymptotic formula.*

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I. Introduction

Most celebrated Weierstrass theorem states that if f is a continuous function on [0.1] then for ε >0, ∃ a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \varepsilon \ , x \in [0,1]$$

S.N. Bernstein [1] gave a constructive proof of the above theorem using the sequence of positive linear operators (Bernstein operators). His polynomial operator $B_n(f; x)$ is defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

There were some drawbacks associated with the operators one of which was slow rate of convergence and the other one, Bernstein operator was not suitable for approximating functions which were integrable in the sense of lebesgue or Riemann.

In those contexts, in 1930, Kantorovich [2] introduced an operator $K_n(f; x)$ defined by

$$K_n(f;x) = (n+1) \sum_{k=0}^{n+1} \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt$$

Where f(x) is an integrable function on [0,1]

A lots of generalizations and modifications of Kantorovich operator have been introduced by many researchers [4-10] with different basis and parameters.

Recently Dhawal J. Bhatt et al. [11] defined Kantorovich type operator with Beta basis function as follows :

For $f \in R[0,1], x \in [0,1]$

$$B_n^*: R[0,1] \to R[0,1]$$

Then
$$B_n^*(f;x) = (n+1) \sum_{r=0}^{n} P_{n,r}(x) \int_{\frac{r}{n+1}}^{\frac{r+1}{n+1}} f(t)dt \qquad (1)$$

where
$$P_{n,r}(x) = \binom{n}{r} \frac{B(nx+r+1, 2n-r-nx+1)}{B(nx+1, n-nx+1)}$$

Where B (a b) is the Beta function defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt \quad (a,b>0)$$

Now we generalize the operator (1) with associating real parameters $(0 \le \alpha \le \beta)$ consequencing the above and defined generalized form as

$\tilde{B}_n : R\{0,1\} \to R[0,1]$ such that

$$\tilde{B}_n(f;x) = (n+\beta+1) \sum_{r=0}^{n} P_{n,r}(x) \int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}} f(t)dt$$

where $f \in R[0,1]$ and $x \in [0,1]$ and

$$P_{n,r}(x) = \binom{n}{r}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)}$$

$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ $(a,b > 0)$ is the Beta function.

It is obvious from the construction that the operator $\tilde{B}_n$ is a positive linear operator.

II. Auxilary Results

Here, we have the following lemma, provides some basic results to derive subsequent results.

Lemma 1 : For two given real parameters α and β, $0 \leqslant \alpha \leqslant \beta$, $a, b > 0$ and non-negative integers n, r we have

(i) For $0 \leqslant r \leqslant n$ $\sum_{r=0}^{n}\binom{n}{r}\frac{B(a+r,b+n-r)}{B(a,b)} = 1$

(ii) For $1 \leqslant r \leqslant n$ $\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(a+r,b+n-1)}{B(a,b)} = \frac{a}{a+b}$

(iii) For $1 \leqslant r \leqslant n$ $\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(a+r,b+n-1)}{B(a,b)}.\frac{r}{n} = \frac{(n-1)(a+1)a}{n(a+b+1)(a+b)} + \frac{a}{n(a+b)}$

(iv) For $1 \leqslant r \leqslant n$ $\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(a+r,b+n-r)}{B(a,b)}\cdot\frac{r^2}{n^2} = \frac{(n-1)(n-2)(a+2)(a+1)a}{n^2(a+b+2)(a+b+1)(a+b)} + \frac{3(n-1)(a+1)a}{n^2(a+b+1)(a+b)} + \frac{a}{n^2(a+b)}$

(v) For $1 \leqslant r \leqslant n$ $\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(a+r,b+n-r)}{B(a,b)}\cdot\frac{r^3}{n^3} = \frac{(n-1)(n-2)(n-3)(a+3)(a+2)(a+1)a}{n^3(a+b+3)(a+b+2)(a+b+1)(a+b)} + \frac{6(n-1)(n-2)(a+2)(a+1)a}{n^3(a+b+2)(a+b+1)(a+b)} + \frac{7(n-1)(a+1)a}{n^3(a+b+1)(a+b)} + \frac{a}{n^3(a+b)}$

Now, in the following lemma, we derive the central moment of $\tilde{B}_n(\mathrm{f};x)$.

Lemma 2. : For x ∈ $[0,1]$, the central moments (moment about origin) of $\tilde{B}_n$ are given by

(i) $\tilde{B}_n(1;x) = 1$

Proof: We have

$$\tilde{B}_n(1;x) = (n+\beta+1)\sum_{r=0}^{n} P_{n,r}(x)\int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}} 1\cdot dt$$

$$= \sum_{r=0}^{n} P_{n,r}(x)$$

From (i) lemma 1, we get

$$\tilde{B}_n(1;x) = 1$$

(ii) $\tilde{B}_n(t;x) = \frac{n}{n+\beta+1}\left(\frac{nx+1}{n+2}\right) + \frac{1+2\alpha}{2(n+\beta+1)}$

Proof: We have

$$\tilde{B}_n(1;x) = (n+\beta+1)\sum_{r=0}^{n} P_{n,r}(x)\int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}} t\cdot dt$$

$$= \frac{1}{n+\beta+1}\left[\sum_{r=0}^{n} P_{n,r}(x)\cdot r + \alpha\sum_{r=0}^{n} P_{n,r}(x)\right] + \frac{1}{2(n+\beta+1)}\sum_{r=0}^{n} P_{n,r}(x)$$

$$= \frac{n}{n+\beta+1}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)}$$

$$+\frac{\alpha}{n+\beta+1} + \frac{1}{2(n+\beta+1)}$$

from (ii) of Lemma 1, we get

$$\tilde{B}_n(t;x) = \frac{n}{n+\beta+1}\left(\frac{nx+1}{n+2}\right) + \frac{1+2\alpha}{2(n+\beta+1)}$$

$$(iii)\ \ \tilde{B}_n\ \ (t^2;x) = \frac{n(n-1)(nx+2)(nx+1)}{(n+\beta+1)^2(n+3)(n+2)} + \frac{2n(nx+1)(1+\alpha)}{(n+\beta+1)^2(n+2)} + \frac{1+3\alpha+\alpha^2}{3(n+\beta+1)^2}$$

Proof: $\tilde{B}_n(t^2;x) = (n+\beta+1)\sum_{r=0}^{n} P_{n,r}(x)\int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}} t^2 dt =$

$$\frac{n^2}{(n+\beta+1)^2}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)}\cdot\frac{r}{n}$$

$$+$$

$$\frac{n(1+2\alpha)}{(n+\beta+1)^2}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)} +$$

$$\frac{1+3\alpha+3\alpha^2}{3(n+\beta+1)^2}$$

from (iii) of the lemma 1

$$\tilde{B}_n(t^2;x) = \frac{n(n-1)(nx+2)(nx+1)}{(n+\beta+1)^2(n+3)(n+2)} + \frac{2\,\mathrm{n}(nx+1)\,(1+\alpha)}{(n+\beta+1)^2(n+2)} + \frac{1+3\alpha+\alpha^2}{3(n+\beta+1)^2}$$

$$\text{(iv) } \tilde{B}_n(t^3;x) = \frac{n(n-1)(n-2)(nx+1)(nx+2)(nx+3)}{(n+\beta+1)^3(n+4)(n+3)(n+2)} + \frac{3(3+2\alpha)n(n-1)(nx+1)(nx+2)}{2(n+\beta+1)^3(n+2)(n+3)} + \frac{n(nx+1)(7+12\alpha+6\alpha^2)}{2(n+\beta+1)^3(n+2)} + \frac{1+4\alpha+6\alpha^2+4\alpha^3}{4(n+\beta+1)^3}$$

Proof:

$$\tilde{B}_n(t^3;x) = (n+\beta+1)\sum_{r=0}^{n} \mathrm{P}_{n,r}(x)\int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}} t^3 dt$$

$$= \frac{1}{4(n+\beta+1)^3}\sum_{r=0}^{n} P_{n,r}(x)\ \begin{bmatrix}4r^3+6(1+2\alpha)r^2 \\ +4(1+3\alpha+3\alpha^2)r \\ +1+4\alpha+6\alpha^2+4\alpha^3\end{bmatrix}$$

$$= \frac{n^3}{(n+\beta+1)^3}\sum_{r=0}^{n} P_{n,r}(x)\frac{r^3}{n^3} + \frac{3n^2(1+2\alpha)}{2(n+\beta+1)^3}\sum_{r=0}^{n} \mathrm{P}_{n,r}(x)\frac{r^2}{n^2} + \frac{n(1+3\alpha+3\alpha^2)}{(n+\beta+1)^3}\sum_{r=0}^{n} P_{n,r}(x)\frac{r}{n} + \frac{1+4\alpha+6\alpha^2+4\alpha^3}{4(n+\beta+1)^3}\sum_{r=0}^{n} P_{n,r}(x)$$

$$= \frac{n^3}{(n+\beta+1)^3}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)}\frac{r^2}{n^2} + \frac{3\,n^2(1+2\alpha)}{2(n+\beta+1)^3}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)}\frac{r}{n} + \frac{n(1+3\alpha+3\alpha^2)}{(n+\beta+1)^3}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)} + \frac{1+4\alpha+6\alpha^2+4\alpha^3}{4(n+\beta+1)^3}$$

Now, from lemma 1 we get the result

(v) $\tilde{B}_n(t^4;x) =$

$$\frac{n(n-1)(n-2)(n-3)(nx+4)(nx+3)(nx+2)(nx+1)}{(n+\beta+1)^4(n+5)(n+4)(n+3)(n+2)} + \frac{8\mathrm{n}(1+\alpha)(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}x+3)(\mathrm{n}x+2)(\mathrm{n}x+1)}{(n+\beta+1)^4(n+4)(n+3)(n+2)} + \frac{3n(5+10\alpha+2\alpha^2)(n-1)(nx+2)(nx+1)}{(n+\beta+1))^4(n+3)(n+2)} + \frac{6n[1+3\alpha+2\alpha^2](nx+1)}{(n+\beta+1)^4(n+2)} + \frac{(1+5\alpha(1+2\alpha+2\alpha^2+\alpha^3))}{5(n+\beta+1)^4}$$

proof: $\tilde{B}_n(t^4;x) = (n+\beta+1)\sum_{r=0}^{n} P_{n,r}(x)\int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}} t^4 dt$

$= \frac{n^4}{(n+\beta+1)^4}\sum_{r=0}^{n} P_{n,r}(x)\frac{r^4}{n^4}+\frac{2n^3(1+4\alpha)}{(n+\beta+1)^4}\sum_{r=0}^{n} P_{n,r}(x)\frac{r^3}{n^3}+ \frac{2n^2(1+3\alpha+3\alpha^2)}{(n+\beta+1)^4}\sum_{r=0}^{n} P_{n,r}(x)\frac{r^2}{n^2}+ \frac{n(1+4\alpha+6\alpha^2)}{(n+\beta+1)^4}\sum_{r=0}^{n} P_{n,r}(x)\frac{r}{n}+\frac{1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4}{5(n+\beta+1)^4}$

$$= \frac{n^4}{(n+\beta+1)^4}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)}\frac{r^3}{n^3}$$
$$+\frac{2n^3(1+4\alpha)}{(n+\beta+1)^4}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)}\frac{r^2}{n^2}$$
$$+\frac{2n^2(1+3\alpha+3\alpha^2)}{(n+\beta+1)^4}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)}\frac{r}{n}$$
$$+\frac{n(1+4\alpha+6\alpha^2)}{(n+\beta+1)^4}\sum_{r=1}^{n}\binom{n-1}{r-1}\frac{B(nx+r+1,2n-r-nx+1)}{B(nx+1,n-nx+1)}$$
$$+\frac{1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4}{5(n+\beta+1)^4}$$

Using the results of lemma 1, we obtain the desired result. Now, for local approximation, the pointwise moments about $x \in [0,1]$ are estimated in the following lemma.

Lemma 3 : for $x \in [0,1]$, nth (n=0,1) moments for the operator $\tilde{B}_n$ about any point x are given by

(i) $\tilde{B}_n((t-x)^0;x) = 1$

Proof : $\tilde{B}_n((t-x)^\circ;x) = \tilde{B}_n(1;x)$

From (i) of lemma 2, we have

$$\tilde{B}_n((t-x)^\circ;x) = 1$$

(ii) $\tilde{B}_n((t-x);x) = \frac{n(3+2\alpha)-2nx(\beta+3)-4x(1+\beta)+2(1+2\alpha)}{2(n+\beta+1)(n+2)}$

Proof : $We\ know\ that$ $\tilde{B}_n$ is monotone and linear

$\tilde{B}_n\big((t-x);x\big) = \tilde{B}_n(t;x) - x$

From (ii) of lemma 2

$$\tilde{B}_n\big((t-x);x\big)=\frac{n(nx+1)}{(n+\beta+1)(n+2)}+\frac{1+2\alpha}{2(n+\beta+1)}-x$$
$$=\frac{n(3+2\alpha)-2nx(\beta+3)-4x(1+\beta)+2(1+2\alpha)}{2(n+\beta+1)(n+2)}$$

(iii) $\tilde{B}_n((t-x)^2;x) = \dfrac{\left[\begin{array}{l} n^3\{6x(1-x)\} + n^2\{x^2(3\beta^2+18\beta+30)- x(18\alpha+6\alpha\beta+9\beta+33)+\alpha^2+9\alpha+13\} \\ + n\{x^2(15\beta^2+60\beta+36)- x(33\beta+66\alpha-30\alpha\beta-51) +5\alpha^2+33\alpha+17\}+18x^2(\beta^2+2\beta) - 18x(1+2\alpha)(1+\beta) \\ +6(1+3\alpha+\alpha^2)\end{array}\right]}{3(n+\beta+1)^2\ (n+3)(n+2)}$

Proof : from monotone property and linearity of $\tilde{B}_n$ and taking the results (ii) and (iii) of lemma2, we obtain the desired result.

Similarly ,we can compute higher order moments of $\tilde{B}_n$ about x using lemma 2.

III. Main Results

In this section, first of all, we establish the uniform convergence of the sequence of $\tilde{B}_n$ to f $\in R[0,1]$ by korovkin's theorem [8].

Here, the norm is defined as

$$\| f \| = \sup_{x\in[0,1]} |f(x)|$$

Theorem 1 : if f $\in R[0,1]$ then

$$\|\hat{B}_n(f;x)-f(x)\| \to 0$$

as n$\to \infty$

Proof : From (i) (ii)and (iii) of lemma 2 ,we have

$$\tilde{B}_n(1;x)=1$$

$$\tilde{B}_n(t;x)=\frac{n}{n+\beta+1}\left(\frac{nx+1}{n+2}\right)+\frac{1+2\alpha}{2(n+\beta+1)}$$

and

$$\tilde{B}_n(t^2;x)=\frac{n(n-1)(nx+2)(nx+1)}{(n+\beta+1)^2(n+3)(n+2)}+\frac{2n(nx+1)(1+\alpha)}{(n+\beta+1)^2(n+2)}+\frac{1+3\alpha+\alpha^2}{3(n+\beta+1)^2}$$

It is obvious that $\tilde{B}_n(t^k;x)$ converges uniformly to x^k,(k = 0,1,2) for $x \in [0,1]$.
Hence by Korovkin's theorem ,the sequence of operators $\tilde{B}_n$ converges uniformly to f $\in C[0,1]$
Now, the following theorem estimates the rate of convergence of the sequence of operators $\tilde{B}_n$,in the terms of modulus of continuity of f $\in C[0,1]$ where the modulus of continuity is given by

$$\omega_f(\delta)\equiv\omega(f,\delta)=\sup_{\substack{x-\delta\le t\le x+\delta\\ a\le x\le b}}|f(t)-f(x)|,\delta>0$$

Theorem 2: If $f\in C[0,1]$ then

$$\left|\tilde{B}_n(f;x)-f(x)\right|\leqslant 2\omega_f\left(\tilde{\delta}_n(x)\right)$$

$$\text{Where } \tilde{\delta}_n(x)=\left(\tilde{B}_n((t-x)^2;x)\right)^{\frac{1}{2}}$$

Proof : We have

$$\left|\tilde{B}_n(f;x)-f(x)\right|\leqslant(n+\beta+1)\sum_{r=0}^{n}P_{n,r}(x)\int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}}|f(t)-f(x)|dt$$

$\omega_f(\delta)$ is an increasing function of δ so, |t-x| $\le \delta$
then

$$|f(t)-f(x)|\leqslant\left(1+\frac{(t-x)^2}{\delta^2}\right)\omega_f(\delta)\leqslant(n+\beta+1)\sum_{r=0}^{n}P_{n,r}(x)\int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}}\left(1+\frac{1}{\delta^2}(t-x)^2\right).\omega_f(\delta)\,dt$$

$$=\ [1+\frac{(n+\beta+1)}{\delta^2}\sum_{r=0}^{n}P_{n,r}(x)\int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}}(t-x)^2\,dt]\omega_f(\delta)$$

$$=\ [1+\frac{1}{\delta^2}\tilde{B}_n((t-x)^2;x)]\ \omega_f(\delta)$$

$$\text{taking } \delta=\tilde{\delta}_n(x)=\left(\tilde{B}_n((t-x)^2;x)\right)^{\frac{1}{2}}$$

$$\text{then } \left|\tilde{B}_n(f;x)-f(x)\right|\leqslant 2\omega_f\left(\tilde{\delta}_n(x)\right)$$

In the next theorem, we estimate the rate of approximation of the sequence of operators $\tilde{B}_n$ in terms of Lipschitz inequality.
Theorem 3: If $f\in \mathrm{Lip}_M(p)$ *then*

$$\left|\tilde{B}_n(f;x)-f(x)\right|\leqslant M\delta_n^p(x)$$

Where $\delta_n(x)=\left(\tilde{B}_n((t-x)^2;x)\right)^{\frac{1}{2}}$

Proof : from the monotonicity of $\tilde{B}_n$,we have

$$\left|\tilde{B}_n(f;x)-f(x)\right|\leqslant(n+\beta+1)\sum_{r=0}^{n}P_{n,r}(x)\int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}}|f(t)-f(x)|dt$$

Since $f\in \mathrm{Lip}_M(p)$

$$|f(t)-f(x)|\leqslant M|t-x|^p$$

Where

$$M>0, 0<p\leqslant 1 \text{ and } t,x\in[0,1]$$

Hence,

$$\left|\tilde{B}_n(f;x) - f(x)\right| \leqslant M \cdot (n+\beta+1) \sum_{r=0}^{n} P_{n,r}(x) \int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}} |t-x|^p dt$$

by Holder's Inequality, we obtain

$$\left|\tilde{B}_n(f;x) - f(x)\right| \leqslant M \cdot \left((n+\beta+1) \textstyle\sum_{r=0}^{n} P_{n,r}(x) \int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}} (t-x)^2 dt\right)^{\frac{p}{2}}$$

$$. \left((n+\beta+1) \textstyle\sum_{r=0}^{n} P_{n,r}(x) \frac{1}{(n+\beta+1)} \right)^{1-\frac{p}{2}}$$

$$= M \left((n+\beta+1) \textstyle\sum_{r=0}^{n} P_{n,r}(x) \int_{\frac{r+\alpha}{n+\beta+1}}^{\frac{r+\alpha+1}{n+\beta+1}} (t-x)^2 \, dt \right)^{\frac{p}{2}}$$

$$= M \left(\tilde{B}_n((t-x)^2;x)\right)^{\frac{p}{2}}$$

$$= M\delta_n^p(x)$$

Where, $\delta_n(x) = \left(\tilde{B}_n((t-x)^2;x)\right)^{\frac{1}{2}}$

IV. Voronovskaya type theorem

Under this Section, now we estimate the asymptotic behavior of the sequence of operators $\tilde{B}_n$ in terms of Voronovskaya type asymptotic formula.

We use the results of the following lemma in subsequent theorem.

Lemma 4: If $x \in [0,1]$, we have

(i) $\lim\limits_{n\to\infty} n \cdot \tilde{B}_n\big((t-x);x\big) = \frac{(3+2\alpha)-2x(3+\beta)}{2}$

(ii) $\lim\limits_{n\to\infty} n \cdot \tilde{B}_n((t-x)^2;x) = 2x(1-x)$

Proof of the above lemma is explicit by using lemma 3.

Theorem 4 : For $f \in C[0,1]$ and $f', f'' \in C[0,1], x \in [0,1]$, we have

$$\lim_{n\to\infty} n \cdot \left(\tilde{B}_n(f;x) - f(x)\right) = \frac{(3+2\alpha)-2x(3+\beta)}{2} f'(x) + x(1-x)f''(x)$$

Proof: From Taylor's formula, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + h(t)(t-x)^2$$

where h(t) is Peano type remainder,continuous function on $[0,1]$ and $\lim\limits_{t\to x} h(t) = 0$

then

$$n\left(\tilde{B}_n(f;x) - f(x)\right) = n \cdot f'(x)\tilde{B}_n\big((t-x);x\big) + \frac{n}{2} f''(x)\tilde{B}_n((t-x)^2;x) + n \cdot \tilde{B}_n(h(t)(t-x)^2;x)$$

From Cauchy-Schwarz inequality

$$\tilde{B}_n(h(t)(t-x)^2;x) \leqslant \sqrt{\tilde{B}_n(h^2(t);x)}\sqrt{\tilde{B}_n((t-x)^4;x)}$$

then

$$n\tilde{B}_n(h(t)(t-x)^2;x) \leqslant \sqrt{\tilde{B}_n(h^2(t);x)}\sqrt{n^2\tilde{B}_n((t-x)^4;x)}$$

refer to lemma 3, it can be seen easily that

$$\lim_{n\to\infty} n^2 \tilde{B}_n((t-x)^4;x) \geqslant 0 \text{ and finite}$$

Now $h(t) \in C[0,1]$, $\lim\limits_{t\to x} h(t) = 0$

and $\tilde{B}_n$ is uniformly convergent for h(t) $\in C[0,1]$

hence

$$\lim_{n\to\infty} \tilde{B}_n(h^2(t);x) = h^2(x) = 0$$

now, we get

$$\lim_{n\to\infty} n \cdot \tilde{B}_n(h(t)(t-x)^2;x) = 0$$

finally, we have

$$\lim_{x\to\infty} n\tilde{B}_n\big((f;x)-f(x)\big)=\frac{(3+2\alpha)-2x(3+\beta)}{2}f'(x)+x(1-x)f''(x)$$

□

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