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A remainder form generated by Cauchy, Lagrange and Chebyshev formulas

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Abstract: *The paper provides preliminary work for obtaining suitable polynomials with a verified value of approximating the error.*
Ideally, the residual term gives you the exact difference between the function value and the approximation $P_n(x)$. However, since the value of c is uncertain, the remaining term actually provides a worst-case scenario of your approximation.
The investigation focuses on the remaining three methods of inferring the value of the constant c; One is based on a Cauchy representation and the other uses the Lagrangian and Chebyshev polynomial basis. We compare the quality of the residues obtained and the performance of the methods with that provided by Taylor's models with the new c.
The natural idea is to try to study the three methods and introduce a new model for estimating the rest of the optimizer and thus can estimate a polynomial suitable for a job.
Keywords: *Error, Remainder, Lagrange's Remainder, Chebyshev Remainder, Cauchy's Remainder*

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I. Introduction:

A common problem in mathematics is that we don't actually know how to calculate certain functions. For example, it's not too easy to calculate $\cos x$ - while we can look up its value in a table, or a graph of the function, these solutions aren't ideal. To solve this problem, mathematicians like to develop very good approximations to these functions - related functions which are very close to the function of interest, but much easier to calculate.

Probably the easiest functions to calculate are the polynomials. It would be nice, then, if we could approximate functions like cosine with Taylor polynomial, Lagrange polynomial or other methods that allow us to do just that.

Many applications appear recently in various areas, e.g., biological systems, chemical reactions, intelligent transportation, just to name a few (good surveys can be found in [1, 6].

Performing numerical computations, yet being able to provide rigorous mathematical statements about the obtained result, is required in many domains like global optimization, ODE solving or integration. Taylor models, which associate to a function a pair made of a Taylor polynomial approximation and a rigorous remainder bound, are a widely used rigorous computation tool.

Taylor model of order n for a function f which is supposed to be $n+1$ times continuously differentiable over an interval $[a,b]$, is a rigorous polynomial approximation of f.

In specific words if the couple(P,ϵ), where P is a polynomial of order(n), and ϵ is an interval part such that $|P(x)-f(x)|<\epsilon,\ \forall x\in[a,b]$. The Taylor polynomial can be generated of the function f. ϵ provides an enclosure of all the approximation errors encountered (truncation, rounding's).

The interpolation polynomial itself is easy to compute and a wide variety of methods and various basis representations exist [12]. Another advantage of interpolation polynomials is that a closed formula, hence an explicit bound, for the remainder exists [9].

Chebyshev polynomial forms are special classes of polynomials especially suited for approximating other functions. They are widely used in many areas of numerical analysis; uniform approximation, least-squares approximation, numerical solution of ordinary and partial differential equations and so on [1].

Chebyshev interpolation polynomial of degree n, roughly speaking, compared to a Taylor remainder, the interpolation error will be scaled down by a factor of 2^{-n}n. For bounding the remainder, the only remaining difficulty is to bound the term $f^{n+1}(c)$ for $c\in I$. However, the advantage of using directly this formula for the

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remainder can not be effectively maintained if the interval bound is obtained using automatic differentiation because of the overestimation [4].
Hence, in what follows the work will try to adapt the constant c approach used by the remainder to get an accuracy value of the remainder one using interpolation polynomials, similar to [5].
If $P_n \in C[a,b]$, an approximation of the function f, it could be generated by the Lagrange interpolating polynomial [1].
Naively thinking that, as more nodes are considered, the approximation will always be more accurate, but this is not always true. The main question to be addressed is whether the polynomials P_n that interpolate a continuous function f in $n+1$ equally spaced points are such that

$$\lim_{n\to\infty}\|f-P_n\| = \lim_{n\to\infty}\|R_n\| = 0,$$

where, if f is sufficiently differentiable, the error can be estimated a sort form.

Definition: *$q \in P_n$ is the best - minimax - polynomial approximation to f on$[a,b]$ if*

$$\|f-q\| \le \|f-p\| \quad \forall p \in P_n.$$

Minimax polynomial approximations exist and are unique when f is continuous, although they are not easy to compute in general. Instead, it is a more effective approach to consider near-minimax approximations, based on Chebyshev polynomials [1].

Lagrange's polynomial approximation:
Lagrange interpolation is a process to interpolate a polynomial to approximate a function f. If $f \in C^{n+1}[a,b]$ and that $x_0, x_1, \dots, x_n \in [a,b]$, then

$$f(x) = P_n(x) + R_n(x),$$

where $P_N(x)$ is a polynomial that can be used to approximate $f(x)$:

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x). \quad (1)$$

where

$$L_{n,k}(x) = \left[\prod_{\substack{j=0\\ j\neq k}}^{n}(x-x_j)\right] \Big/ \left[\prod_{\substack{j=0\\ j\neq k}}^{n}(x_k-x_j)\right].$$

The error term has the form

$$R_n(x) = \frac{Q(x)}{(N+1)!} f^{(n+1)}(c),$$

where $Q(x) = (x-x_0)(x-x_1)\dots(x-x_n)$, and for some c lies between x_0 and x , $x_i \neq x$, $i = 0,1,2,\dots,n$, [8].
If f is defined on $[a,b]$ that contains equally spaced nodes $x_k = x_0 + hk$, and f is bounded in sub intervals $[x_k, x_{k+1}]$, then it could be noted that

$$\left|f^{(n+1)}(x)\right| \le M, \quad (2)$$

for some $M > 0$, then

$$|R_n| \le \frac{Q(x)}{(N+1)!} M,$$

Theorem: *Let f be n times continuously differentiable on the interval $[x_0, x]$, and let $f^{(n+1)}$ exists in the open interval (x_0, x). Then, for some $c \in (x_0, x)$,*

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

The remainder $f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ is said to be in Lagrange's form [10].

Cauchy's remainder:
The Cauchy remainder after a nth term of the Taylor series for a function f expanded about a point x_0 is given by

$$R_n = \frac{f^{(n+1)}(c)}{n!}(x-x_0)\,(x-c)^n,$$

where $c \in [x_0, x]$.

Note that, the Cauchy remainder $R_n$is also sometimes taken to refer to the remainder when terms up to the n power are taken in the Taylor series [10], see also: Lagrange Remainder, Schlomilch's Remainder, Taylor's Inequality, Taylor Series.

Theorem (Remainder Estimation Theorem): *Suppose the (n + 1)st derivative $f^{(n+1)}(x)$ exists for all x in some interval I contains* 0. *For all $x \in I$,*

$$|R_n(x)| \leq \frac{M|x|^{n+1}}{(n+1)!},$$

where M is the maximum value of $f^{(n+1)}$ in the interval I.
That the Cauchy and Lagrange forms of the remainder are equal is not something you need to show. Since they are both expressions for the same value, they are equal (however- the unknown constants in the forms have not the same value).

$$|R_n(x)| = \frac{f^{(n+1)}(c)}{n!}(x-a)(x-c)^n.$$

The problem is to prove this by the Cauchy version. For simplicity, it is assumed it is about 0 so,

$$|R_n(x)| = \frac{f^{(n+1)}(c)}{n!}(x)(x-c)^n,$$

for some c between 0 and x,
using the statement (2); $\left|f^{(n+1)}(x)\right| \leq M$ for all x and n, then

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}|x-c|^n|x| < \frac{M}{n!}|x|^n|x| = \frac{M|x|^{n+1}}{n!},$$

which goes to 0 as $n \to \infty$.
Note that, since c is between x and 0, we have $|x-c| < |x|$.
So $\quad |R_n(x)| \leq \frac{M|x-x_0|^{n+1}}{(n+1)!}, \quad \forall x \in I.$

It is a variant of the Taylor-Lagrange formula. If the function f is with real values and that it is differentiable on I up to the order $n+1$ then, for all $x \in I \backslash \{x_0\}$, there is a number c strictly between x_0 and x such that

$$|R_n(x)| = \frac{f^{(n+1)}(c)}{n!}(x-x_0)(x-c)^n,$$
$$|R_n(x)| \leq \frac{M(x-x_0)(x-c)^n}{n!}.$$

Chebyshev polynomials interpolation:

Chebyshev polynomials form is a special class of polynomials especially suited for approximating other functions. They are widely used in many areas of numerical analysis, uniform approximation, least-squares approximation, numerical solution of ordinary and partial differential equations, and so on [1].
Chebyshev polynomial over $[-1,1]$ can be defined as

$$T_n(x) = \cos(n\, anccos\, x), \qquad x \in [-1,1], \qquad n > 0.$$

$T_n(x)$ satisfice the properties:

$$T_n(x) = 2xT_n(x) - T_{n-1}(x), \qquad n = 1, 2, ...,$$
$$T_0(x) = 1, \qquad T_1(x) = x,$$
$$T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x.$$

The leading coefficients of x^n in $T_n(x)$ is 2^{n-1} and $T_n(-x) = (-1)^n T_n(x)$.
$T_n(x)$ has n zeros which lie in the interval $(-1,1)$, that are given by

$$x_k = \cos\left(\frac{2k+1}{2n}\pi\right), \quad k = 0, 1, \dots, n-1.$$

Since the functions are considered over any interval $I = [a, b]$, then the Chebyshev polynomials are defined as

$$T_n^{(a,b)}(x) = T_n\left(\frac{2x-b-a}{b-a}\right),$$

$T_n^{(a,b)}$ has $n+1$ distinct real roots in $[a, b]$ 'Chebyshev nodes', since they are of utmost interest for interpolation:

$$x_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{(i+1/2)\pi}{n+1}\right), \quad i = 1, 2, \dots, n.$$

Definition: A polynomial P of degree n is called monic if the coefficient of x^n is 1.

Theorem: *If P is a monic polynomial of degree n, then*

$$\|P(x)\|_\infty \equiv max_{-1<x<1}|P(x)| \geq 2^{1-n}.$$

Minimax approximation:
One might first try to directly use the minimax polynomial that is the polynomial which minimizes the supremum norm of the approximation error. In fact, such a polynomial can be obtained only through an iterative procedure.
One upper bound for $|R_n|$ can formed by taking the product of the maximum value of $|Q(x)|$ over all $x \in [-1,1]$ and the maximum value of $\left|f^{(n+1)}(x)/(n+1)!\right|$ over all $x \in [-1,1]$, Chebyshiv discovered that $x_0, x_1, \dots, x_n$ should be chosen so that $Q(x) = (\frac{1}{2^n})T_{n+1}(x)$ [9].

Theorem *The polynomial $\bar{T}_n(x) = 2^{(1-n)}T_n(x)$ is the minimax approximation on $[-1,1]$ to the zero function by a monic polynomial of degree n and*

$$\|\bar{T}_n(x)\| = 2^{1-n}.$$

Theorem: *If the nodes x_i are chosen as the roots of the Chebyshev polynomial*

$$T_{n+1}(x),\ x_i = cos\left(\frac{2i+1}{2n+2}\pi\right), \quad (i = 0, 1, 2, \dots, n).$$

then the error term for polynomial interpolation using the nodes x_i is

$$R_n(x) = |f(x) - P(x)| \le \frac{1}{2^n(n+1)!} max_{-1<x<1}\left|f^{(n+1)}(t)\right|.$$

Moreover, this is the best upper bound could be achieved by varying the choice of the x_i.

Theorem (Chebyshev Approximation):
The Chebyshev approximation polynomial $P_n(x)$ of degree $\le n$ for $f(x)$ over $[-1,1]$ can be written as a sum of $\left(T_j(x)\right)$ [8].

$$f(x) \approx P_n(x) = \sum_{i=0}^{n} c_i T_i(x). \qquad (3)$$

Where the coefficients $\{c_i\}$are computed by the function

$$c_0 = \frac{1}{n+1}\sum_{k=0}^{n} f(x_k)\,T_0(x_k) = \frac{1}{n+1}\sum_{k=0}^{n} f(x_k),$$

and

$$c_j = \frac{2}{n+1}\sum_{k=0}^{n} f(x_k)\,T_j(x_k),$$

$$= \frac{2}{n+1}\sum_{k=0}^{n} f(x_k)\cos\left(\frac{(2k+1)j\pi}{2(n+1)}\right), \quad \text{for } j = 1, 2, \dots, n.$$

By turning the attention to polynomial interpolation for $f(x)$ over $[-1,1]$ based on the nodes $-1 \le x_0 \le x_1 \le \cdots \le x_n \le 1$.
Both the Lagrange, Cauchy and Chebyshev polynomials satisfy

$$f(x) = P_n(x) + R_n(x), \quad x \in [-1,1]\,,$$

where the remainder term is $R_n(x)$.
Using the relationship

$$|R_n(x)| \le \frac{\max\{|f^{(n+1)}(x)|\}}{(n+1)!}\max\{|Q(x)|\}, \quad x \in [-1,1].$$

The task is to determine how to select the set of nodes $\{x_k\}_{k=0}^{n}$ that minimizes $\max\{|Q(x)|\}$. Research investigating the minimum error in polynomial interpolation is attributed to the Russian mathematician Pafnuty Lvovich Chebyshev (1821-1894).
The error term for polynomial interpolation using the nodes x_i is

$$|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|\,\bar{T}_{n+1}(x)| \le 2^{-n}\frac{|f^{(n+1)}(c)|}{(n+1)!} \le \frac{1}{2^n(n+1)!}\left\|f^{(n+1)}(c)\right\|.$$

By using the equation (2) which gives $\left|f^{(n+1)}(t)\right| \le M$, the remainder will be as:

$$|R_n(x)| \le \frac{1}{2^n(n+1)!}M. \qquad (4)$$

Moreover, this is the best upper bound could be achieved by varying the choice of the x_i.
In the Chebyshev approximation, the average error can be large but the maximum error is minimized. Chebyshev approximations of a function are sometimes said to be minimax approximations of the function.
A pathological example for which the Lagrange interpolation does not converge is provided by $f(x) = |x|$ in the interval $[-1,1]$, for which equidistant interpolation diverges for $0 < |x| < 1$.
A less pathological example, studied by Runge, showing the *Runge phenomenon*, gives a clear warning on the problems of equally spaced nodes.

Theorem: (Lagrange-Chebyshev Approximation) *Assume that $P_n(x)$ is the Lagrange polynomial that is based on the Chebyshev interpolating nodes $\{x_k\}_{k=0}^{n}$ on [a,b] mentioned above.*
If $f(x) \in C^{n+1}[a,b]$ then

$$|f(x) - P_n(x)| \leq \frac{2(b-a)^{n+1}}{4^{n+1}(n+1)!} max_{a\leq x\leq b}\{|f^{(n+1)}(x)|\},$$

holds for all $x \in [a,b]$.
New form of the remainder:

As it is discussed above, the Cauchy remainder form is

$$|R_{nC}(x)| = \frac{f^{(n+1)}(c)}{n!}(x - x_0)\,(x - c)^n,$$

here $c \in [x_0, x]$.
The Chebyshev remainder form is

$$|R_{nV}(x)| \leq \frac{\|f^{(n+1)}(c)\|}{2^n(n+1)!},$$

these forms can be written as

$$|R_{nC}(x)| \leq \frac{M}{n!}(x - x_0)\,(x - c)^n, \qquad (5)$$

and

$$|R_{nV}(x)| \leq \frac{M}{2^n(n+1)!}, \qquad (6)$$

where $f^{(n+1)}(c) \leq M$.
from (5) and (6)

$$|R_{nC}| \approx |R_{nV}|,$$

then

$$\frac{M}{n!}(x - x_0)\,(x - c)^n \approx \frac{M}{2^n(n+1)!},$$

$$(x - x_0)\,(x - c)^n \approx \frac{1}{2^n(n+1)},$$

$$c \approx x - 1/(2\sqrt[n]{(x - x_0)(n+1)}). \qquad (7)$$

Moreover, it could introduce the previous form to calculate the constant c to estimate the remainder from by Lagrange or Chebyshev forms.
Furthermore, by using Lagrange and Chebyshev remainder forms, it could introduce the following form:

$R_{nL}(x)$(Lagrange remainder) $\leq \frac{\prod_{i=0}^{n}(x-x_i)M}{(n+1)!}$, (8)

$R_{nV}(x)$(Chebyshev remainder) $\leq \frac{M}{2^n(n+1)!}$, (9)

Where $c \in [x_0, x]$, and $M \geq f^{(n+1)}(c)$.
The form (8) could be modified into the form

$$|R_{nL}(x)| \approx \frac{(x-c)\prod_{i=0}^{n-1}(x-x_i)M}{(n+1)!}, \qquad (10)$$

x_n is substituted by c, from (9) and (10), we get

$$\frac{M}{2^n(n+1)!} \approx \frac{(x-c)\prod_{i=0}^{n-1}(x-x_i)M}{(n+1)!},$$

$$c = x - \frac{1}{2^n\prod_{i=0}^{n-1}(x-x_i)}, \qquad (11)$$

By using the form (7) or (11), the magic value c can be appreciated which gives a good estimate of the error to generate occasion polynomial function.

Example:
Consider the function $f(x) = \cos x$, $x \in [-1,1]$, find an equivalent polynomial to the function $f(x)$ and the remainder at point $x = 0.5$, using the Lagrange and the Chebyshev processes, and compare the results .

1- Lagrange polynomial:

Let us divide the interval $[-1,1]$ into 4 points; x_0, x_1, x_2, x_3, where

$$x_0 = -1,\ x_1 = -0.333,\ x_2 = 0.333,\ x_3 = 1,$$

which give

$$y_0 = 0.540302,\ y_1 = 0.945066,\ y_2 = 0.945066,\ y_3 = 0.540302,$$

According to the equation (1), The Lagrange polynomial is

$$P_{3L}(x) = -0.455245x^2 + 0.99553,$$

and the apparent error is

$$|R_3(0.5)| = |f(0.5) - P(0.5)| = 0.004136,$$

and the remainder as in equation (1) above depends on the value of c, table 1 shows the exact error of the polynomial and tables 2 and 3 show the error according to the value of c.

2- Chebyshev polynomial:

Firstly the Chebyshev nods are

$$x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right), \quad k = 0, 1, 2, 3,$$

	Lagrange Polynomial	Chebyshev Polynomial
$P(x)$	$-0.455245x^2 + 0.99553$	$-0.57228x^2 + 1.028775$
$P_3(0.5)$	0.881736	0.885705
$R_3(0.5)$	0.004136	0.008122

$$x_0 = 0.92388, \quad x_1 = 0.382683, \; x_2 = -0.382683, x_3 = -0.92388$$

and

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \;\; T_3(x) = 4x^3 - 3x$$

so, the Chebyshev polynomial according to the equation (3) is

$$P_{3V}(x) = -0.466784x^2 + 0.998589,$$

the apparent error is

$$|R_3(0.5)| = |f(0.5) - P(0.5)| = 0.00431,$$

Table 1: The appeared Lagrange and Chebyshev polynomials and their remainders at $x = 0.5$.

The Chebyshev remainder depends on the value of c, the results of the study are contained in the tables 1, 2 and 3, see also figures 1, 2.

c	$R_{3C}(0.5)$	$R_{3L}(0.5)$	$R_{3V}(0.5)$
0.0	0.03125	0.004347	0.005208
0.1	0.01592	0.004325	0.00518
0.2	0.00661	0.00426	0.005105
0.3	0.00191	0.004153	0.004976
0.3486	**0.00081**	**0.0041**	**0.0049**
0.4	0.00023	0.004004	0.004797
0.5	0.00	0.003815	0.004571

Table 2: Cauchy's, Lagrange's and Chebyshev's remainder at different values of c.

Now using the new form of the constant $c \in [-1, 0.5]$ prescriped in equation (6);

$$c \approx x - 1/(2\sqrt[n]{(x - x_0)(n + 1)}),$$

then

$$c \approx 0.5 - 1/\left[2\sqrt[3]{(0.5 + 1)(3 + 1)}\right] = 0.3486,$$

so, the remainder using the new c is;

$$|R_{3L}| = \frac{Q(x)f^{(n+1)}(c)}{(n+1)!} = \frac{(0.104333)(0.974839)}{4!} = 0.0041,$$

$$|R_{3C}| = \frac{0.975}{6}(0.5 + 1)(0.5 - 0.2248)^3 = 0.00082,$$

and

$$|R_{3V}| = \frac{f^{(n+1)}(c)}{2^n(n+1)!} = \frac{0.974839}{2^3 * 4!} = 0.0049.$$

n	c	$R_{nL}(0.5)$	$R_{nC}(0.5)$	$R_{nV}(0.5)$
1	0.3333	0.0493	0.2362	0.2362
2	0.3333	0.0164	0.0197	0.0394
3	**0.3486**	**0.0041**	**0.00082**	**0.0049**
4	0.3635	0.000813	0.00002	0.000486

Table 3 shows Lagrange's, Cauchy's and Chebyshev's remainders at different values of n and c according to equation (7).

n	c	$R_{nL}(0.5)$	$R_{nC}(0.5)$	$R_{nV}(0.5)$
1	-1.8962	0.167	1.1489	0.0799
2	-0.6981	0.0133	0.8247	0.0319
3	**-0.099**	**0.0043**	**0.0535**	**0.0052**
4	0.2005	0.00085	0.00049	0.00051

Table 4 the Lagrange's, Cauchy's and Chebyshev's remainders at different values of n and c according to equation (11).

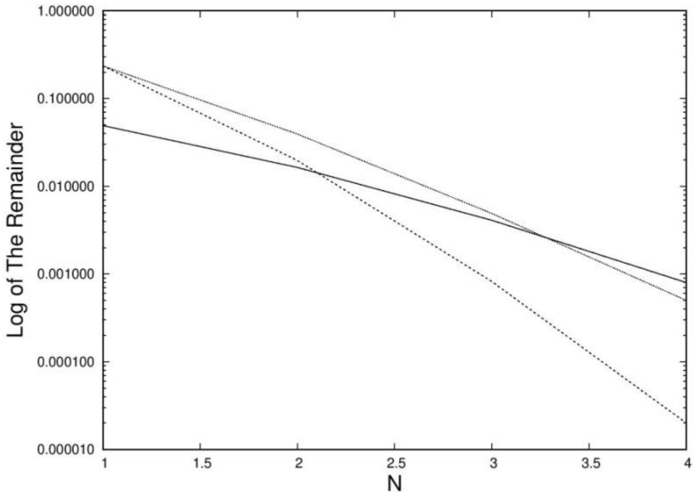

Figure 1: The remainders of Cauchy dark line, Lagrange long dashed line and Chebyshev short dashed line vs n.

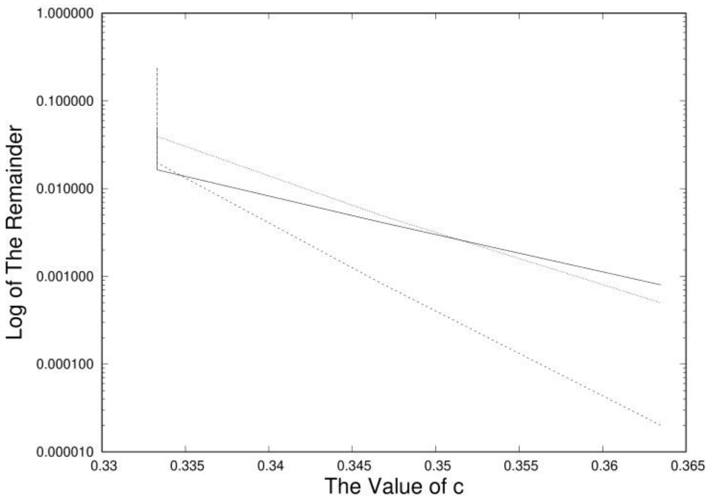

Figure 2: the remainders of Cauchy dark line, Lagrange long dashed line and Chebyshev short dashed line vs the value of c

II. Conclusion:

From the above, the study has been established to deduce the magic number c from which it is possible to verify the remainder R_n. In addition, it helps in building a suitable polynomial. Moreover, it gives the degree of polynomial appropriate to the function. In this way it is possible to predict at least the degree of the polynomial and any method that achieves the required accuracy, in this way the time and effort can be reduced in constructing a polynomial that simulates the original function. The resulting c value can be used with Lagrange and Chebyshev polynomials in addition to Cauchy, and in all cases gives good results as shown in the given example. In addition, it was noticed that the method of the rest of the Chebyshev did not depend on the given x-value and therefore was constant for all x-value, the rest of the Lagrange was more accurate for a number of decimal places, while the Cauchy method had less accurate results.

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