

The Non-Existence Of Perfect Odd Numbers

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Abstract: The aim of this article is to solve a problem opened more than 2 millennia ago (300 BC) : There is no odd perfect number.

Résumé : Le but de cet article est de résoudre un problème ouvert il y'a plus de 2 millénaires (300 av. J.-C) à savoir qu'il n'existe aucun nombre **parfait impair**.

Key words: perfect number, number theory.

Date of Submission: 28-04-2021

Date of Acceptance: 12-05-2021

I. Introduction :

A perfect number ([1,3]) is a natural number equal to half of the sum of its divisors or is the sum of its proper divisors (Example : $6=1+2+3$, $12=1+2+3+6$). Pythagoras then Euclid the discovered more than 2 millennia ago (300 BC), but we ignores almost everything about them.

Despite the simplicity of the definition, two essential questions not yet found an answer from Euclid. The first: is there a infinity of perfect numbers? The second: are there numbers perfect odd?

Recently, fourteen researchers from Brigham Young University in Provo (Utah, United States) have uploaded to the open access site arXiv [2] an article giving an account of new insights and ways of research concerning the second riddle, whether there are numbers perfect odd - even if they admit that the resolution is still " out of range "[1].

The purpose of this article is to answer their question by proving that there is no perfect odd number.

Theorem: There is no odd perfect number.

Proposition 1: If N is a non-zero natural number $N \in \mathbb{N}^*$ with $N = \prod_{i=1}^n P_i^{\alpha_i}$ and the P_i are prime disjoint two by two, then: N is **perfect** is equivalent to:

$$2 \prod_{i=1}^n P_i^{\alpha_i} = \prod_{i=1}^n (1 + \sum_{j=1}^{\alpha_i} P_i^j)$$

Proof : Deduced from the fact that the term of the line is the sum of all the divisors of N .

Corollary 1:

Let $N \in 2\mathbb{N}^* + 1$. If N is perfect then $N = \prod_{i=1}^n P_i^{\alpha_i}$ with:

- The P_i are disjoint prime numbers two by two.
- $\alpha_1 \in 2\mathbb{N}^* + 1$
- $\alpha_i \in 2\mathbb{N}$ if $i \neq 1$

Proof: It results from the **proposition 1** above and the fact that $N \in 2\mathbb{N}^* + 1$ (odd).

Corollary 2:

Let $M \in 2\mathbb{N}^* + 1$. If $N = M^2$ then N is not perfect.

Proof : It results from the **Corollary 1**.

Proof of the theorem:

Theorem: There is no odd perfect number.

Proof :

Let N be an **odd** integer.

If N is **perfect**, then from the **corollaries** 1 and 2 we deduce that: $N = p^{2k+1} M^2$, with p a prime number not dividing M.

N is assumed to be perfect, therefore:

$$N = p^{2k+1} M^2 = \sum_i d_i(M^2) + (1 + \sum_{j=1}^{2k+1} p^j) + \sum_{j=1}^{2k+1} p^j (\sum d_i(M^2)) + M^2 + \sum_{j=1}^{2k} p^j M^2 .$$

With $d_i(M^2)$ a divisor of M^2 distinct from 1 and M^2 .

We deduce that in $\mathbb{Z} / p\mathbb{Z}$ we have : $0 = \sum_i d_i(M^2) + 1 + M^2$.

Let $\sum_i d_i(M^2) + 1 + M^2 = p^l K$ with $p \nmid K = 1$.

First K is odd, because if it is even, from the equality $\sum_i d_i(M^2) + 1 + M^2 = p^l K$ with $p \nmid K = 1$, we deduce that 2 divides $\sum_i d_i(M^2)$, so $\sum_i d_i(M^2)$ is even.

But M^2 has an odd number of divisors (because it is a square), moreover the divisors of M^2 are all odd, so $\sum_i d_i(M^2)$ must be odd. Which is contradictory.

By substituting in the above equality, we will have:

$$p^{2k+1} M^2 = p^l K + \sum_{j=1}^{2k+1} p^j + \sum_{j=1}^{2k+1} p^j (\sum d_i(M^2)) + \sum_{j=1}^{2k} p^j M^2$$

$$\text{So } p^{2k+1} M^2 = p^l K + \sum_{j=1}^{2k+1} p^j + \sum_{j=1}^{2k+1} p^j (p^l K - 1 - M^2) + \sum_{j=1}^{2k} p^j M^2$$

$$\text{And } p^{2k+1} M^2 = p^l K + p \frac{p^{2k+1} - 1}{p - 1} + p \frac{p^{2k+1} - 1}{p - 1} (p^l K - 1 - M^2) + p \frac{p^{2k} - 1}{p - 1} M^2$$

$$\text{We deduce that: } p^{2k} M^2 = p^{l-1} K + \frac{p^{2k+1} - 1}{p - 1} + \frac{p^{2k+1} - 1}{p - 1} (p^l K - 1 - M^2) + \frac{p^{2k} - 1}{p - 1} M^2$$

$$\text{And } p^{2k} M^2 = p^{l-1} K + \frac{p^{2k+1} - 1}{p - 1} (p^l K - M^2) + \frac{p^{2k} - 1}{p - 1} M^2$$

$$\text{And finally } 2 p^{2k} M^2 = p^{l-1} K + p^l K \frac{p^{2k+1} - 1}{p - 1} = p^{l-1} K (1 + p \frac{p^{2k+1} - 1}{p - 1})$$

Since K is odd, then $l - 1 = 2k$. And we will have:

$$2 M^2 = K (1 + p \frac{p^{2k+1} - 1}{p - 1}) = K (\frac{p^{2k+2} - 1}{p - 1}) = K (\frac{p^{k+1} - 1}{p - 1}) (p^{k+1} + 1)$$

If $p \equiv 3 \pmod{4}$, from the above equality, we deduce that $2 M^2 \equiv 0 \pmod{4}$, and M and therefore N will be even, which is impossible.

The remaining and most delicate case is the one where $p \equiv 1 \pmod{4}$ which I will demonstrate by the following stratagem:

From the equality :

$$N = p^{2k+1} M^2 = \sum_i d_i(M^2) + (1 + \sum_{j=1}^{2k+1} p^j) + \sum_{j=1}^{2k+1} p^j (\sum d_i(M^2)) + M^2 + \sum_{j=1}^{2k} p^j M^2 , \text{ we deduce:}$$

$$M^2 = \frac{\sum d_i(M^2)}{p^{2k+1}} + \frac{1}{p^{2k+1}} + \frac{\sum_{j=1}^{2k+1} p^j}{p^{2k+1}} + \frac{\sum_{j=1}^{2k+1} p^j}{p^{2k+1}} (\sum d_i(M^2)) + \frac{\sum_{j=1}^{2k} p^j}{p^{2k+1}} M^2 + \frac{M^2}{p^{2k+1}}$$

So :

$$M^2 = \sum d_i(M^2) \frac{\left(\frac{1}{p}\right)^{2k+2} - 1}{\frac{1}{p} - 1} + \frac{1}{p^{2k+1}} + \frac{1}{p^{2k}} \frac{p^{2k+1} - 1}{p - 1} + \frac{M^2 \left(\frac{1}{p}\right)^{2k+1} - 1}{p \frac{1}{p} - 1}$$

And :

$$M^2 \left(1 - \frac{1}{p} \frac{\left(\frac{1}{p}\right)^{2k+1} - 1}{\frac{1}{p} - 1}\right) - \left(\sum d_i(M^2)\right) \frac{\left(\frac{1}{p}\right)^{2k+2} - 1}{\frac{1}{p} - 1} = \frac{1}{p^{2k+1}} + \frac{1}{p^{2k}} \frac{p^{2k+1} - 1}{p - 1}$$

As $1 - \frac{1}{p} \frac{\left(\frac{1}{p}\right)^{2k+1} - 1}{\frac{1}{p} - 1} = \frac{\left(\frac{1}{p}\right)^{2k+2} - 1}{\frac{1}{p} - 1}$, Then we have :

$$M^2 - \sum d_i(M^2) = \frac{\frac{1}{p} - 1}{\left(\frac{1}{p}\right)^{2k+2} - 1} \left(\frac{1}{p^{2k+1}} + \frac{1}{p^{2k}} \frac{p^{2k+1} - 1}{p - 1}\right) \leq 2 \left(\frac{1}{p^{2k+1}} + \frac{1}{p^{2k}} \frac{p^{2k+1} - 1}{p - 1}\right) \leq 2(1/5+2) \leq 5$$

So : $M^2 = \sum d_i(M^2) + \mu$ with $\mu \in \{0, 2, 4\}$ (because $M^2 - \sum d_i(M^2)$ is an even integer), but N is perfect, so :

$$\begin{aligned} N &= p^{2k+1} M^2 \\ &= p^{2k+1} \left(\sum d_i(M^2) + \mu\right) \\ &= p^{2k+1} + \sum_{j=0}^{2k} p^j + p^{2k+1} \left(\sum d_i(M^2)\right) + \left(\sum d_i(M^2)\right) \left(\sum_{j=0}^{2k} p^j\right) + \left(\sum_{j=0}^{2k} p^j\right) M^2 \end{aligned}$$

$$\text{So : } (\mu - 1) p^{2k+1} = \left(\sum_{j=0}^{2k} p^j\right) (1 + M^2 + \sum d_i(M^2)) .$$

Let us distinguish two cases:

If $k \neq 0$, we have an immediate contradiction by using the above equality.

If $k = 0$: by using the equality :

$$M^2 - \sum d_i(M^2) = \frac{\frac{1}{p} - 1}{\left(\frac{1}{p}\right)^{2k+2} - 1} \left(\frac{1}{p^{2k+1}} + \frac{1}{p^{2k}} \frac{p^{2k+1} - 1}{p - 1}\right)$$

We deduce that $M^2 - \sum d_i(M^2) = 1$ which is absurd (because $M^2 - \sum d_i(M^2)$ is an even integer).

Hence the Theorem.

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Mohamed SGHIAR. "The Non-Existence Of Perfect Odd Numbers." *IOSR Journal of Mathematics (IOSR-JM)*, 17(3), (2021): pp. 49-51.