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On Transforming the Laplace Operator

Henrik Stenlund

Abstract
In this communication it is shown that a function of the Laplace operator acting on an arbitrary function can be transformed to a three-dimensional integral. The cases of the exponential function and of an arbitrary function expressible as a power series, are treated. Two special cases of radial functions are presented based on elementary observations made here. 1

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I. Introduction

The Laplace operator is a bit troublesome to manage in various expressions including differential equations. Difficulties add up when the Laplace is an argument of a function. As an important example, the exponential function is treated in the following

$$b(\bar{r}) = e^{\alpha\nabla^2}c(\bar{r}) \tag{1}$$

The exponential operator is defined as a power series and all practical operations with it are based on that. The exponential expression is more of a symbolic nature. It would be even more useful for many applications that a function of the Laplace could be transformed to some other form which is more manageable. This is typical for development of wave equations and solutions of the Cauchy problem, for instance in [3], [4]. Various diffusion problems may also benefit from these results. On one occasion, this has already been made while solving the linear diffusion in three dimensions [5]. Otherwise, this topic is studied only relatively little recently [1], [2].

In the following, transforming the exponential Laplace is solved based on Fourier transform in three dimensions. The generalization to a function of a Laplace operator is presented next. In addition, some series expressions are given for special cases of radial functions utilizing some results obtained in the first part. Any formal proofs of the solutions and methods are omitted for clarity.

II. Transforming the Exponential Laplace Operator

The exponential Laplace operator will be treated in the following to give it a form suitable for further processing. We assume the function c(**r**) is Fourier integrable scalar function and may be complex-valued.

α

is a complex constant in general.

$$b(\bar{r}) = e^{\alpha\nabla^2}c(\bar{r}) \tag{2}$$

By using the three-dimensional Fourier transform of c(**r**) one can see that

$$b(\bar{r}) = e^{\alpha\nabla^2} \int \frac{d\bar{k}}{(2\pi)^{\frac{3}{2}}} e^{i\bar{k}\cdot\bar{r}} \tilde{c}(\bar{k}) \tag{3}$$

From vector analysis it becomes clear that since the

$$\nabla^2$$

acts above only on the coordinate **r**, one will get

$$\nabla^2 e^{i\bar{k}\cdot\bar{r}} = -k^2 e^{i\bar{k}\cdot\bar{r}} \tag{4}$$

which is not too hard to prove. Therefore the exponential operator can be expressed as

$$e^{\alpha\nabla^2} e^{i\bar{k}\cdot\bar{r}} = \sum_{n=0} \frac{\alpha^n (\nabla^2)^n}{n!} e^{i\bar{k}\cdot\bar{r}} = \sum_{n=0} \frac{\alpha^n (-k^2)^n}{n!} e^{i\bar{k}\cdot\bar{r}} \tag{5}$$

yielding

$$e^{\alpha\nabla^2} e^{i\bar{k}\cdot\bar{r}} = e^{-\alpha k^2} e^{i\bar{k}\cdot\bar{r}} \tag{6}$$

Therefore

$$b(\bar{r}) = \int \frac{d\bar{k}}{(2\pi)^{\frac{3}{2}}} e^{i\bar{k}\cdot\bar{r} - k^2\alpha} \tilde{c}(\bar{k}) \tag{7}$$

By reinserting the original transform one will get

$$b(\bar{r}) = \int \frac{d\bar{r'} c(\bar{r'})}{(2\pi)^{\frac{3}{2}}} \int \frac{d\bar{k}}{(2\pi)^{\frac{3}{2}}} e^{-i\bar{k}\cdot\bar{r'} + i\bar{k}\cdot\bar{r} - k^2\alpha} \tag{8}$$

The **k** integral is elementary and can be calculated to give

$$b(\bar{r}) = e^{\alpha\nabla^2} c(\bar{r}) = \frac{1}{(4\pi\alpha)^{\frac{3}{2}}} \int d\bar{r'} e^{\frac{-|\bar{r} - \bar{r'}|^2}{4\alpha}} c(\bar{r'}) \tag{9}$$

The exponential operator has thus been converted to a three-dimensional integral over the whole volume of the function's domain. This result is known and derived by Fourier series producing the heat kernel.

III. Generalizing the Exponential Laplace Operator

One can follow the steps of the preceding chapter while generalizing the exponential function to an arbitrary function of the Laplace. Other assumptions are the same as above and one will use some of the results obtained there.

$$b(\bar{r}) = f(\nabla^2) c(\bar{r}) \tag{10}$$

The three-dimensional Fourier transform of c(**r**) is used to get

$$b(\bar{r}) = f(\nabla^2) \int \frac{d\bar{k}}{(2\pi)^{\frac{3}{2}}} e^{i\bar{k}\cdot\bar{r}} \tilde{c}(\bar{k}) \tag{11}$$

Now it is assumed that the function f() has a MacLaurin series expansion which is a fair assumption for many functions.

$$f(z) = \sum_{n=0} \frac{f_n z^n}{n!} \tag{12}$$

This expression sends a signal of caution since the argument z is here a differential operator. The function should not contain dependence of **r** in fn in such a way that it is compromising the operator. The reason is the non-commutability of these items and this will create huge difficulties in expanding the MacLaurin series. Any spatial dependence is preferred to be on the left side of the operator. For instance, the function below would be very complicated to handle while acting on a function.

$$f(\nabla^2) = e^{r^2 \alpha \nabla^2 \frac{1}{r}} \tag{13}$$

When it is expanded as a power series of terms

$$(r^2 \alpha \nabla^2 \frac{1}{r})^n \tag{14}$$

and operating on a function on the right, there will result uncomfortable cross terms.

The series expansion should also be converging, but as an operator expansion, that is nearly impossible to prove. In order to determine the convergence of the series, the terms should be studied while the operator is acting on the target function itself.

$$f(\nabla^2) c(\bar{r}) = \sum_{n=0} \frac{f_n (\nabla^2)^n}{n!} c(\bar{r}) \tag{15}$$

Thus the target function becomes part of the convergence, which is a natural requirement. If the operator function can not be expanded as a converging power series, then the formal results may be false, leading to invalid expressions. The convergence must be valid throughout the infinite domain of the argument as required by the Fourier integrals. A Taylor series is applicable here too as long as the binomial does not contain any spatial dependence. The end result will be the same.

By applying the operator function to the exponential function, one is getting

$$f(\nabla^2) e^{i\bar{k}\cdot\bar{r}} = \sum_{n=0} \frac{f_n \cdot (\nabla^2)^n}{n!} e^{i\bar{k}\cdot\bar{r}} = \sum_{n=0} \frac{f_n \cdot (-k^2)^n}{n!} e^{i\bar{k}\cdot\bar{r}} \tag{16}$$

yielding

$$f(\nabla^2)e^{i\bar{k}\cdot\bar{r}} = f(-k^2)e^{i\bar{k}\cdot\bar{r}} \qquad (17)$$

It is assumed that the power series will converge. Thus

$$b(\bar{r}) = \int \frac{d\bar{k}}{(2\pi)^{\frac{3}{2}}} e^{i\bar{k}\cdot\bar{r}} f(-k^2)\tilde{c}(\bar{k}) \qquad (18)$$

By reinserting the original transform one will get

$$b(\bar{r}) = \int \frac{d\bar{r'}}{(2\pi)^{\frac{3}{2}}} \int \frac{d\bar{k}}{(2\pi)^{\frac{3}{2}}} e^{-i\bar{k}\cdot\bar{r'}+i\bar{k}\cdot\bar{r}} f(-k^2)c(\bar{r'}) \qquad (19)$$

This will turn into

$$b(\bar{r}) = f(\nabla^2)c(\bar{r}) = \int \frac{d\bar{r'}c(\bar{r'})}{(2\pi)^{\frac{3}{2}}} \int \frac{d\bar{k}}{(2\pi)^{\frac{3}{2}}} e^{i\bar{k}\cdot(\bar{r}-\bar{r'})} f(-k^2) \qquad (20)$$

The operator function is converted to a double three-dimensional integral over the whole volume of the function's domain acting on the target function. The inner integral is actually a three-dimensional Fourier transform of

$$f(-k^2).$$

Other requirements on the function f() are mild in addition to the aforementioned requirements on convergence of the power series. It is obvious that the first exponential case (9) is produced from this equation. This result is believed to be new. As a further generalization the function c(**r**) can be a vector function whose component functions are subject to the same requirements as above. This forces the b(**r**) to be a vector function as well.

3.1 A Simple Example of a Function

As a one-dimensional example is shown the following. The result in the preceding section can be flattened to one dimension as

$$f(\frac{\partial^2}{\partial x^2})c(x) = \int_{-\infty}^{\infty} \frac{dx'c(x')}{2\pi} \int_{-\infty}^{\infty} dke^{ik(x-x')} f(-k^2) \qquad (21)$$

Flattening is done by observing that there is no dependence on y, z and the inner integrals become Dirac delta functions. The example is the following operator acting on the function c()

$$e^{b\frac{\partial}{\partial x}} c(x) \qquad (22)$$

This can be seen as a function

$$e^{b\sqrt{\frac{\partial^2}{\partial x^2}}} c(x) \qquad (23)$$

Thus one will have

$$f(-k^2) = e^{ibk} \tag{24}$$

$$e^{b\frac{\partial}{\partial x}}c(x) = \int_{-\infty}^{\infty} \frac{dx'c(x')}{2\pi} \int_{-\infty}^{\infty} dke^{ik(x-x'+b)} \tag{25}$$

The latter integral becomes a delta function too, leaving

$$e^{b\frac{\partial}{\partial x}}c(x) = \int_{-\infty}^{\infty} dx'c(x')\delta(x-x'+b) = c(x+b) \tag{26}$$

This is a familiar result from applying the differential operator in equation (22).

IV. Series expressions in Some Special Cases

4.1 Action of the Exponential Laplace on a Power of Radius

In the following, some expressions are developed giving alternate forms for the exponential operator. The exponential is expanded as a power series and the target function is a power of the radius r.

$$e^{\beta\nabla^2}r^k = \sum_{n=0}^{\infty} \frac{\beta^n(\nabla^2)^{n-1}}{n!}\nabla^2 r^k \tag{27}$$

From vector analysis it is well known that

$$\nabla^2 r^k = (k+1)kr^{k-2} \tag{28}$$

Using this one can obtain the following

$$e^{\beta\nabla^2}r^k = \sum_{n=0}^{\infty} \frac{\beta^n}{n!}(k+1)k(k-1)(k-2)...(k-2n+2)r^{(k-2n)} \tag{29}$$

The factorial function can be taken into use

$$x^{(m)} = x(x-1)(x-2)...(x-(m-1)) \tag{30}$$

with

$$x = k+1, m = 2n-1 \tag{31}$$

Further, the factorial function has a representation in terms of Stirling's numbers of first kind

$$x^{(m)} = \sum_{1}^{m} S_{1i}^{m} x^i \tag{32}$$

giving

$$e^{\beta\nabla^2}r^k = r^k(1 + \sum_{n=1}^{\infty} \frac{\beta^n r^{-2n}}{n!} \sum_{i=1}^{2n} S_{1i}^{2n-1}(k+1)^i) \tag{33}$$

4.2 Action of the Exponential Laplace on a Radial Function

The result obtained in the preceding section can be extended to a function of radius r. Assuming the function can be expanded as a MacLaurin power series, one can have

$$e^{\beta\nabla^2}h(r)=\sum_{k=0}^{\infty}\frac{h_k}{k!}e^{\beta\nabla^2}r^k \qquad (34)$$

The hk are the series coefficients. The preceding section gives immediately

$$e^{\beta\nabla^2}h(r)=\sum_{k=0}^{\infty}\frac{h_k}{k!}r^k(1+\sum_{n=1}^{\infty}\frac{\beta^n r^{-2n}}{n!}\sum_{i=1}^{2n}S_{1i}^{2n-1}(k+1)^i) \qquad (35)$$

V. Conclusions

This paper presents a few results regarding both the exponential Laplace operator and an arbitrary function of it. The equation (9) shows how to change an exponential Laplace operator to an integral operator. To perform this operation, Fourier analysis in three dimensions is applied in a straightforward way. The result, which is known as the heat kernel, is applicable in solving wave equations and diffusion equations. The method by which it is reached, is new. The generalization of this method leads to equation (20) for an arbitrary function. This is believed to be a new result. Thus it is possible to transform a function of the Laplace operator to an integral expression which is actually a double three-dimensional integral. The inner integral is a Fourier transform of the operator function with an argument

$$-k^2$$

One requirement for the functions affected with the Laplace is that they are Fourier transformable. Another requirement is that the operator function expressed as a power series acting on the target function does exist and converges in the whole argument domain. The function must be of such a form that any spatial dependence does not interfere with the operator but remains on the left side of it while operating on the target function to the right. Else cross terms will appear and things get very complicated, preventing the use of equation (20).

While applying equation (20) for the case of a plain Laplace

$$f(\nabla^2)=\nabla^2 \qquad (36)$$

the integrals can be solved but, unfortunately, the solution will loop back to the Laplace itself.

The following two, equations (33) and (35) treat specific cases of radial functions subjected to the exponential Laplace. They apply somewhat different approaches delivering expressions applicable in numeric and symbolic processing.

References

[1]. Rosenbeg, S.: *The Laplacian on a Riemannian Manifold: An Introduction to Analysis on Manifolds* Cambridge University Press (**1997**)

[2]. Mikhail Belkin, Jian Sun, Yusu Wang: *Constructing Laplace Operator from Point Clouds in Rd* **Proceedings of the twentieth annual ACM-SIAM**, 2009 - SIAM siam.org (**2009**)

[3]. Stenlund, H.: *On Solving the Cauchy Problem with Propagator Series*, **arXiv:1411.1402v1**[math.AP] 5 Nov (**2014**)

[4]. Stenlund, H.: *A Closed-Form Solution to the Arbitrary Order Cauchy Problem with Propagators*, **arXiv:1411.6890v1**[math.AP] 24 Nov (**2014**)

[5]. Stenlund, H.: *On the Basics of Linear Diffusion*, **IOSR Journal of Mathematics (IOSR-JM)** e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 2 Ser. III (Mar. { Apr. **2020**), PP 44-49 www.iosrjournals.org

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