A Study of the Beam Fixed At Ends and Loaded In the Middle and Cantilever

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Abstract: A cantilever is a thin unvaried cross-sectional beam unchanging horizontally at one end and loaded at the additional end. In this paper, we talk about the theory of the cantilever and the beam supported at its ends and loaded in the centervia the application of residue theorem to findtotal depression at a given point on them and it comes out to be very effective tool for analyzing the theory of the cantilever and the beam supported at its ends and loaded in the middle for finding the total depression at a given point on their length.

Keywords: Residue Approach, Cantilever, Beam, Mahgoub Transformation

I. Introduction:

The Residue Theorem approach has been apply for analyzing most of the problems in science and engineering [1, 2, 3, 4] like analysis of electric circuits, analysis of quantum theory of free electron gas, integral equations of convolution type, etc. It also comes out to be very powerful mathematical tool for analyzing the theory of cantilever and the beam supported at its ends and loaded in the middle to find their total depression at a given point on them. A structure of uniform cross-section, whose length is large in comparison to its breadth and thickness, is called as beam. For such a structure, the shearing stress over any cross-section is negligibly small. A cantilever is a thin uniform beam fixed horizontally at one end and loaded at the other end [5]. Beams are used in the construction of bridges or for supporting the heavy loads, and are commonly used in the structure of multistoried buildings [6,7,8,9,10,11,12]. The Mahgoub Transform has been applied in solving boundary value problems in most of the science and engineering disciplines. The electrical circuits are usually analyzed by adopting different integral transforms or methods [13,1415,16,17,18,19,20]. In this paper, we will study a Mahgoub integral analyze the impulsive response of network circuits and reveals that it can also be obtained easily by the application of Mahgoub Transform.

II. Definitions

2.1 Mahgoub Transform

If the function $f(y)$, $y \geq 0$ is having an exponential order and is a piecewise continuous function on any interval, then the Mahgoub transform of $\dot{h}(y)$ is given by

$$ M\{\dot{h}(y)\} = \hat{h}(p) = p \int_{0}^{\infty} e^{-py} \dot{h}(y) dy. $$

The Mahgoub Transform [1, 2, 3] of some of the functions are given by

- $M\{y^n\} = \frac{n!}{p^{n+1}}$, where $n = 0,1,2,..$
- $M\{e^{ay}\} = \frac{p}{p-a}$
- $M\{sinay\} = \frac{ap}{p^2+a^2}$
- $M\{cosay\} = \frac{p^2}{p^2+a^2}$
- $M\{sinhay\} = \frac{ap}{p^2-a^2}$
- $M\{coshay\} = \frac{p^2}{p^2-a^2}$

2.2 Inverse Mahgoub Transform

The Inverse Mahgoub Transform of some of the functions are given by

- $M^{-1}\{\frac{1}{p^n}\} = \frac{y^n}{n!}, n = 2,3,4 ...$
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- $M^{-1}\left(\frac{p^2}{p^2-a^2}\right) = e^{ay}$
- $M^{-1}\left(\frac{p}{p^2+a^2}\right) = \frac{1}{a}\sin ay$
- $M^{-1}\left(\frac{p}{p^2+a^2}\right) = \cos ay$
- $M^{-1}\left(\frac{ap}{p^2+a^2}\right) = \frac{1}{a}\sin hay$
- $M^{-1}\left(\frac{p}{p^2-a^2}\right) = \frac{1}{a}\cos hay$

2.3 Mahgoub Transform of Derivatives

The Elzaki Transform [1, 2, 3] of some of the Derivatives of $h(y)$ are given by
- $M[h'(y)] = pM[h(y)] - p \cdot h(0)$
- $M[h''(y)] = p^2\hat{h}(p) - p^2\hat{h}(0) - p\hat{h}'(0)$

Residue theorem

It states [1, 2] that $y(x) = \sum$ of residues of $e^{px}\tilde{y}(p)$ at the poles of $\tilde{y}(p)$, where $\tilde{y}(p)$ is the Laplace transform of $y(x)$.

Methods

The important methods [1, 4, 8] of finding residues are
(i) If $y(x)$ has a simple pole at $x = b$, then residue of $y(x)$ at $x = b$ i.e. $\text{Res}(b) = \lim_{x\to b}(x - b)y(x)$.
(ii) Residue (for pole of order $n$) $= \lim_{x\to b}rac{1}{n-1!}\frac{d^{n-1}}{dx^{n-1}}[x^n y(x)]$

III. Methodology

CANTILEVER

Assume a horizontal beam CD of length $L$, fixed at the end C and loaded with load $W$ vertically downwards at the free end D. The bending moment [5, 6] at the section X is given by the differential equation

$\frac{d^2}{dx^2}y(x) + \frac{W}{YI}(L - x) = 0$... (a),

where $I$ is the geometrical moment of inertia and $Y$ is the Young’s modulus. Here $y$ is the depression of the beam at the section X at the distance $x$ from fixed end $A$.

On taking the Mahgoub Transform [9, 10] of (1), we get

$p^2\tilde{y}(p) - p^2y(0) - py(0) + \frac{W}{YI}(L - \frac{1}{p^2}) = 0$... (b)

Here $\tilde{y}(p)$ is the mahgoub transform of $y(x)$.

Put $y(0) = 0$, $\tilde{y}(0) = 0$ therefore, (2) becomes,

$p^2\tilde{y}(p) + \frac{W}{YI}(L - \frac{1}{p^2}) = 0$

Or

$\tilde{y}(p) = -\frac{W}{YI}(Lp - 1) \frac{1}{p^3}$

The pole of $\tilde{y}(p)$ is 0 and it is of order four. Residue of $e^{px}\tilde{y}(p)$ at $p = 0$ is equal to

$\lim_{p\to 0} \frac{1}{2!}\frac{d^2}{dp^2}[e^{px} p^3 \tilde{y}(p)]$

On solving, we get

Residue of $e^{px}\tilde{y}(p)$ at $p = 0$ is $\frac{W}{2YI}(2Lx - x^2)$.

Applying Residue Theorem, we have

$y(x) = \text{Sum of residues of } e^{px}\tilde{y}(p) \text{ at its poles i.e.}$

$y(x) = \frac{1}{2YI}\left[2W(2Lx - x^2)\right]$... (c)

At $x = L$ i.e. at the free end, the total depression is given by

$y(L) = \frac{1}{2YI}\left[2WL^2 - L^2\right]$... (d)

For a beam of circular cross-section [12, 13]

$I = \pi r^4/4$
Hence from (4), we have

\[ y(L) = \frac{2WL^2}{Y\pi r^4} \]

**BEAM SUPPORTED AT ENDS AND LOADED IN THE MIDDLE**

Consider a beam supported on the two knife edges C and D and loaded in the middle with a load W vertically downwards. Assume L be the length of the beam between the points A and B. The bending moment \([5, 6]\) at the section X is given by the differential equation

\[ \ddot{y}(x) + \frac{W}{2YI} x = 0 \quad (e) \]

where I is the geometrical moment of inertia and Y is the Young’s modulus. Here y is the depression of the beam at the section X at the distance x from the end A.

On taking the Laplace Transform \([10, 11]\) of (5), we get

\[ p^2 \hat{y}(p) - p \hat{y}(0) - p\dot{y}(0) + \frac{W}{2YI} \frac{1}{p} = 0 \quad (f) \]

Here \(\hat{y}(p)\) is the Laplace transform of \(y(x)\).

Put \(\dot{y}(0) = 0, \ddot{y}(0) = K\) therefore, equation (6) becomes,

\[ p^2 \hat{y}(p) - K + \frac{W}{2YI} \frac{1}{p} = 0 \]

\[ \hat{y}(p) = \frac{Kp - \frac{W}{2YI}}{p^3} \]

The pole of \(\hat{y}(p)\) is 0 and it is of order four. Residue of \(e^{px} \hat{y}(p)\) at \(p = 0\) is equal to

\[ \lim_{p \to 0} \frac{d}{dp} \left[ e^{px} p^3 \hat{y}(p) \right] = \lim_{p \to 0} \frac{d^2}{dp^2} \left[ e^{px} p^3 \left( \frac{Kp - \frac{W}{2YI}}{p^3} \right) \right] \]

On solving, we get

Residue of \(e^{px} \hat{y}(p)\) at \(p = 0\) is \(\frac{1}{2i} (4Kx - \frac{W}{2YI} x^2)\).

Applying Residue Theorem, we have

\[ y(x) = \sum \text{Residues of } e^{px} \hat{y}(p) \text{ at its poles i.e.} \]

\[ y(x) = \frac{1}{2i} (4Kx - \frac{W}{2YI} x^2) \quad (g) \]

At \(x = \frac{L}{2}\), \(\ddot{y}(\frac{L}{2}) = 0\). Therefore, using (g) and solving for K, we get

\[ C = \frac{Wl}{4YI} ... (i) \]

Use (g) in (i), we get

\[ y(x) = \frac{1}{2i} \left[ \frac{WLx}{2YI} - \frac{Wx^2}{2YI} \right] \quad (j) \]

At the middle, the total depression is given by

\[ y(L/2) = \frac{1}{2i} \left[ \frac{WLx}{2YI} - \frac{Wx^2}{2YI} \right] \]

Or

\[ y(L/2) = \frac{WL^2}{16YI} \quad (k) \]

For a beam of circular cross-section \([12, 13]\)

\[ I = \pi r^4 / 4 \]

Hence from (10), we have

\[ y(L/2) = \frac{WL^2}{Y\pi r^4} \]

**IV. Conclusion:**

We have Analysis the theory of the cantilever and the beam supported at its ends and loaded in the middle and found the total depression at the free end of the cantilever, and at the middle of the beam supported at its ends and loaded in the middle via the application of residue theorem. It is found that the depression increases as the lengths of the cantilever and the beam supported at its ends and loaded in the middle increases.
References:


[10]. Theory of structures by R.S. Khurmi, S. Chand And Company LTD.


