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Compatible Maps with Common Fixed Point Theorems in Fuzzy Normed Space

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Abstract. *Popa[6] explored the common fixed point forsemi-compatible maps choosing the family F_4 of implicit real functionsin d-complete topological space. A class of implicit relationis used by Singh and Jain[8] to prove a common fixed point theorem in fuzzy metric space.In fuzzy normed spaces, Singh et al.[7] proved fixed point theoremfor two self-maps and Chauhan etal.[1] proved a result for common fixed point of four self-maps using the concept of compatibility.*

Main result of this paper presents a common fixed point theorem for six self-maps in fuzzy normed space employing a class of implicit relation.The result of Singh et al.[7], Popa[6], Singh and Jain[8] and Chauhan etal.[1] are generalized in this paper.

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I. Introduction.

Fuzzy set was introduced by Zadeh[9] in 1965 and George[2] defined Fuzzy normed space in 1995. Jose and Santiago[3] presented the notion of Fuzzy norm on a real or complex vector space and defined Fuzzy normed space called F-normed space which was the modification of the definition of F- normed spaces given by George[2] andalso defined the convergence of a sequence and F-Cauchy sequence in F-normed space.The idea of compatibility was introduced by Mishra et al.[5]in fuzzy metric space.The notion of weak compatibility was givenJungck [4] which was the generalization of compatibility. Singh and Jain [8] established the concept of semi-compatibility in fuzzy metric space.

Definition 1. [3] A 3-tuple (X, N, *),where X is a vector space which is real or complex, N is a function on $X\times (0, \infty)$ and * is a continuous t-norm is said to be F-normed space ifthe followingconditions are satisfied:

(i) $N(x, t) = 1$ if and only if $x = 0$;
(ii) $N(x, t) > 0$;
(iii) $N(x, t) * N(y, s) \le N(x + y, t + s)$;
(iv) $N(kx, t) = N(x, t/|k|)$;
(v) $N(x, .) : (0, \infty) \to [0, 1]$ is continuous,

for s, t> 0 and for all $x, y \in X$.

Let (X, N, *) be a F-normed space. Define $N(x - y, t) = M(x, y, t)$ for $x, y \in X$, $t > 0$. Then (X, M, *) is a fuzzy metric space.

Definition 2.[3] A sequence $\{x_n\}$ is said to be convergent to an element $x \in X$in a F-normed space (X, N, *) if and only if there exists an $n_0 \in J$ such that $N(x_n - x, t) > 1 - q$, for every $n \ge n_0$, where $t > 0$, $0 < q < 1$.

Definition 3. [3] A sequence $\{x_n\}$is said to be F-Cauchy sequence in a F-normed space (X, N, *) if and only ifthere exists an $n_0 \in J$ such that $N(x_n - x_m, t) > 1 - s$, for every $n, m \ge n_0$, where $t > 0$, $0 < s < 1$.

Definition4. [3] If every F-Cauchy sequence in X converges to an element in X then F-normed space (X, N, *) is said to be complete.

Remark 1. The limit of a sequence is unique in a F- normed space (X, N, *).

Lemma 1.[7] In a F-normed space (X, N, *),sequence $\{y_n\}$ is said to be F-Cauchy if there exists $k \in (0, 1)$ such that

$N(y_n - y_{n+1}, kt) \geq N(y_{n-1} - y_n, t)$ where $t > 0$.

Definition 5.[1]The pair (A, B) of self-maps is said to be compatiblein a F-normed space (X, N, *) if $\lim_{n\to\infty} N(ABx_n - BAx_n, t) = 1$ for all t >0,where $\{x_n\}$ is a sequence in X, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, for some $x \in X$.

Definition 6. The pair (P, B) of self-maps is said to be weakly compatiblein a F-normed space (X, N, *) if they commute at their coincidence point i.e. for some x in X , $N(Px - Bx, t) = 1$ implies $N(PBx - BPx, t) = 1$ that is Px = Bx implies PBx = BPx.

Definition 7. The pair (P, B) of self-maps is said to be semi-compatible in a F-normed space (X, N, *) if $\lim_{n\to\infty} N(PBx_n - Bx, t) = 1$ for all t > 0where $\{x_n\}$ is a sequence in X and $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Bx_n = x$.

Main Result.

Popa[6] proved a result concerning common fixed point of semi-compatible mappings in d-complete topological space using the family F_4 of implicit real functions. Singh et al.[7] presented thefixed point theorems for two self-maps in fuzzy normed space.Chauhan et al. [1] proved a result for four mappings using compatibility in fuzzy normed space. In fuzzy metric space,Singh and Jain[8] established a result for fixed point theoremsusing the following class of implicit relation:

Let Φ be the set of all real continuous functions $\phi : (R^+)^4 \to R$ which are non-decreasing in first argumentfulfilling the conditions:

(a) if $\phi(u, u, 1, 1) \geq 0$ then $u \geq 1$;

(b) If $\phi(u, v, v, u) \geq 0$ or if $\phi(u, v, u, v) \geq 0$ then $u \geq v$ for $u, v \geq 0$.

Here we are going to prove a common fixed point theorem for six self-maps in fuzzy normed space employing a class of implicit relation provided by Singh and Jain [8].The result of Singh et al. [7], Popa [6], Singh and Jain[8] and Chauhan et al. [1] are generalized here.

Theorem. Let (X, N, *) be a complete F-normed space and A, B, P, Q, S and T be self-maps with the conditions:

(1) $PQ(X) \subseteq S(X)$, $AB(X) \subseteq T(X)$;

(2) either AB orS is continuous;

(3) the pair (PQ, T) is weaklycompatibleand the pair (AB, S) is semi-compatible;

(4) PT = TP,PQ = QP, SA = AS and AB = BA;

there exists $c \in (0, 1)$ for $\phi \in \Phi$, such that for t > 0 and for all x, y in X,

(5) $\phi(N(ABx - PQy, ct), N(Sx - Ty, t), N(ABx - Sx, t), N(PQy - Ty, ct)) \geq 0$.

(6) $\phi(N(ABx - PQy, ct), N(Sx - Ty, t), N(ABx - Sx, ct), N(PQy - Ty, t)) \geq 0$.

Then A, B, P, Q, S and T have a unique common fixed point in X.

Proof. Take an arbitrary point x_0 in X. Since $PQ(X) \subseteq S(X)$ and $AB(X) \subseteq T(X)$, then there exists $x_1, x_2 \in X$ such that $PQx_1 = Sx_2$ and $ABx_0 = Tx_1$.Form sequences $\{x_n\}$ and $\{y_n\}$ in X such that $ABx_{2n} = Tx_{2n+1} = y_{2n+1}$ and $PQx_{2n+1} = Sx_{2n+2} = y_{2n+2}$ for n = 0, 1, 2,… .

To prove $\{y_n\}$ is a cauchy sequence in X, substituting x_{2n} for x and x_{2n+1} for y in (5), we obtain

$$\phi(N(ABx_{2n} - PQx_{2n+1}, ct), N(Sx_{2n} - Tx_{2n+1}, t), N(ABx_{2n} - Sx_{2n}, t), N(PQx_{2n+1} - Tx_{2n+1}, ct)) \geq 0.$$

$$\phi(N(y_{2n+1} - y_{2n+2}, ct), N(y_{2n} - y_{2n+1}, t), N(y_{2n+1} - y_{2n}, t), N(y_{2n+2} - y_{2n+1}, ct)) \geq 0.$$

Using (b), $N(y_{2n+2} - y_{2n+1}, ct) \geq N(y_{2n+1} - y_{2n}, t)$.

Similarly, substituting x_{2n+2} for x and x_{2n+1} for y in (6), we obtain

$$\phi(N(y_{2n+3} - y_{2n+2}, ct), N(y_{2n+1} - y_{2n+2}, t), N(y_{2n+3} - y_{2n+2}, ct), N(y_{2n+1} - y_{2n+2}, t)) \geq 0.$$

Using (b), we get$N(y_{2n+3} - y_{2n+2}, ct)) \geq N(y_{2n+1} - y_{2n+2}, t)$.
In general$N(y_n - y_{n+1}, ct) \geq N(y_{n-1} - y_n, t) \ \forall\ t > 0$.

From Lemma 1, $\{y_n\}$ is a Cauchy sequence in X. As (X, N, *) is complete, $\{y_n\}$ converges to some point in X say z. Therefore, subsequences of sequence $\{y_n\}$ which are $\{ABx_{2n}\}$, $\{PQx_{2n+1}\}$, $\{Sx_{2n+2}\}$ and $\{Tx_{2n+1}\}$ converges to z.

Case I. Let us take S is continuous and the pair (AB, S) is semi-compatibile, we have $S(ABx_{2n}) \to Sz$, $S^2x_{2n} \to Sz$ and $AB(Sx_{2n}) \to Sz$.

Step 1. Substituting$Sx_{2n}$for x and $x_{2n+1}$for y in (5), we have

$$\phi(N(ABSx_{2n} - PQx_{2n+1}, ct), N(SSx_{2n} - Tx_{2n+1}, t), N(ABSx_{2n} - SSx_{2n}, t), N(PQx_{2n+1} - Tx_{2n+1}, ct)) \geq 0.$$

If $n \to \infty$, $\phi(N(Sz - z, ct), N(Sz - z, t), N(Sz - Sz, t), N(z - z, ct)) \geq 0$.

$$\phi(N(Sz - z, ct), N(Sz - z, t), 1, 1) \geq 0.$$

In first argument ϕ is non-decreasing, we get $\phi(N(Sz - z, t), N(Sz - z, t), 1, 1) \geq 0$. Using (a),$N(Sz - z, t) \geq 1$ implies$Sz = z$.

Step 2. Substituting z for x and $x_{2n+1}$for y in (5),

$$\phi(N(ABz - PQx_{2n+1}, ct), N(Sz - Tx_{2n+1}, t), N(ABz - Sz, t), N(PQx_{2n+1} - Tx_{2n+1}, ct)) \geq 0.$$

If $n \to \infty$ $\phi(N(ABz - z, ct), 1, N(ABz - z, t), 1) \geq 0$.In first argument ϕ is non-decreasing and using (b), $N(ABz - z, t) \geq 1$ implies $ABz = z$.

Step 3. As $AB(X) \subseteq T(X)$, there exists $v \in X$ such that $z = ABz = Tv$.
Substituting x_{2n} for x and v for y in (5), we have

$$\phi(N(ABx_{2n} - PQv, ct), N(Sx_{2n} - Tv, t), N(ABx_{2n} - Sx_{2n}, t), N(PQv - Tv, ct)) \geq 0.$$

If $n \to \infty$ $\phi(N(z - PQv, ct), 1, 1, N(PQv - z, ct)) \geq 0$.In first argument ϕ is non-decreasing and using (b), $N(z - PQv, t) \geq 1$give$z = PQv$.
As the pair (PQ, T) is weakly compatible, it gives
$TPQv = PQTv$ or $Tz = PQz$.

Step 4. Substituting z for x and y in (5), we have

$$\phi(N(ABz - PQz, ct), N(Sz - Tz, t), N(ABz - Sz, t), N(PQz - Tz, ct)) \geq 0.$$

From previous results $\phi(N(z - PQz, ct), N(z - PQz, t), 1, 1) \geq 0$.

In first argument ϕ is non-decreasing and using (a), $N(z - PQz, t) \geq 1$which implies that$z = PQz$. Therefore$z = ABz = Sz = PQz = Tz$.

Step 5. Substituting z for x and Qz for y in (5), we have

$$\phi(N(ABz - PQQz, ct), N(Sz - TQz, t), N(ABz - Sz, t), N(PQQz - TQz, ct)) \geq 0.$$

As $TQ = QT$ and $PQ = QP$ so that $PQ(Qz) = Qz$and $T(Qz) = Qz$.
From previous results, $\phi(N(z - Qz, ct), N(z - Qz, t), 1, 1) \geq 0$.

In first argument ϕ is non-decreasing, $\phi(N(z - Qz, t), N(z - Qz, t), 1, 1) \geq 0$.
Using (a), $N(z - Qz, t) \geq 1$implies that$z = Qz$.Now $PQz = z$implies that $Pz = z$.

Step 6. Substituting Az for x and z for y in (5), we have

$$\phi(N(ABAz - PQz, ct), N(SAz - Tz, t), N(ABAz - SAz, t), N(PQz - Tz, ct)) \geq 0.$$

As $SA = AS$ and $AB = BA$ so that $SAz = Az$ and $AB(Az) = Az$.
From previous, $\phi(N(Az - z, ct), N(Az - z, t), 1, 1) \geq 0$.

In first argument ϕ is non-decreasing and using (a), $N(Az - z, t) \geq 1$. Therefore$z = Az$.
which implies that $ABz = BAz = Bz = z$.
Hence$Az = Bz = Pz = Qz = Sz = Tz = z$.
Thus, z is a common fixed point of all six self-maps.

Case II. Let AB is continuous, $(AB)Sx_{2n} \to ABz$ and $(AB)ABx_{2n} \to ABz$. Since the pair (AB, S) is semi-compatible implies $(AB)Sx_{2n} \to Sz$. Uniqueness of limit in F-normed space implies that Sz = ABz.

Step 7. Substituting z for x and x_{2n+1} for y in (5)

$$\phi(N(ABz - PQx_{2n+1}, ct), N(Sz - Tx_{2n+1}, t), N(ABz - Sz, t), N(PQx_{2n+1} - Tx_{2n+1}, ct)) \geq 0.$$

If $n \to \infty$, $\phi(N(ABz - z, ct), N(ABz - z, t), 1, 1) \geq 0$. In first argument ϕ is non-decreasing $\phi(N(ABz - z, t), N(ABz - z, t), 1, 1) \geq 0$. Using (a), $N(ABz - z, t) \geq 1$ which implies that ABz = z. Hence Sz = ABz = z. Now, applying steps 3, 4, 5 and 6, we get

$$Az = Bz = Pz = Qz = Sz = Tz = z.$$

Hence z is a common fixed point all six self-maps .

Uniqueness of the fixed point.

Assume that z' is also a common fixed point of all six self-mappings taken in the theorem, therefore $Az' = Bz' = Pz' = Qz' = Sz' = Tz' = z'$.

Substituting z for x and z' for y in (5)

$$\phi(N(ABz - PQz', ct), N(Sz - Tz', t), N(ABz - Sz, t), N(PQz' - Tz', ct)) \geq 0.$$

$$\phi(N(z - z', ct), N(z - z', t), 1, 1) \geq 0.$$

In first argument ϕ is non-decreasing, therefore $\phi(N(z - z', t), N(z - z', t), 1, 1) \geq 0$. Using (a), we obtain $N(z - z', t) \geq 1$, which gives that $z = z'$ and z is unique common fixed point of all six self-maps.

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