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A Few Interesting Sequences

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I. Definition

Let the sequence $\{x_n\}$ be defined by the seed $x_1 = x_2 = 1$, and the recursive relation

$$\boxed{x_{n+2} = x_n + \frac{1}{x_{n+1}}} \quad (1)$$

The first ten terms are

$$1,1,\frac{2}{1},\frac{3}{2},\frac{8}{3},\frac{15}{8},\frac{48}{15},\frac{105}{48},\frac{384}{105},\frac{945}{384} \quad (2)$$

For reasons that will be apparent later, we do not reduce the fractions in (2) to lowest terms. Now multiplying both sides of (1) by x_{n+1} yields

$$x_{n+1}x_{n+2} = x_n x_{n+1} + 1 \quad (3)$$

Letting$f(n) = x_n x_{n+1}$, (3) becomes $f(n + 1) = f(n) + 1$. Since $f(1) = x_1 x_2 = 1$, we find that $f(n) = n$, implying that $x_n x_{n+1} = n$. It follows that

$$\boxed{x_{n+1} = \frac{n}{x_n}} \quad (4)$$

In other words, the sequence defined by the recursive relation (1) in which each term depends on the two terms preceding it has a simpler recursive relation (4) in which each term depends solely on the preceding term. The reader should verify that the first ten terms satisfy the new recursion.

We use (4) to write another recursive formula from which the sequence may be obtained (with the seed $x_1 = x_2 = 1$). We have, using (4),

$$x_{n+2} = \frac{n+1}{x_{n+1}} = \left(\frac{n+1}{n}\right)x_n \Rightarrow \boxed{x_{n+2} = \left(\frac{n+1}{n}\right)x_n} \quad (5)$$

II. Several theorems

As a consequence of (5), we have the following theorem.

Theorem 1: $\boxed{\lim_{n\to\infty}\left(\dfrac{x_{n+2}}{x_n}\right)^n = e}$

Proof: By (5), we have $\left(\dfrac{x_{n+2}}{x_n}\right)^n = \left(1+\dfrac{1}{n}\right)^n$ which approaches *e* as *n* goes to infinity. ■

We now find closed form expressions for the sequence, using (4). The two seed terms will be omitted, but can be shown to follow the pattern to be established. Once again, the fractions in Table 1, below, will not be reduced to lowest terms.

$$x_3 = \frac{2}{x_2} = \frac{2}{1} = \frac{(1!)^2 2^2}{2!}$$

$$x_4 = \frac{3}{x_3} = \frac{1\cdot 3}{2} = \frac{1\cdot 2\cdot 3}{2^2} = \frac{3!}{(1!)^2 2^2}$$

$$x_5 = \frac{4}{x_4} = \frac{2\cdot 4}{1\cdot 3} = \frac{(2\cdot 4)^2}{1\cdot 2\cdot 3\cdot 4} = \frac{(2!)^2 2^4}{4!}$$

$$x_6 = \frac{5}{x_5} = \frac{1\cdot 3\cdot 5}{2\cdot 4} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{(2\cdot 4)^2} = \frac{5!}{(2!)^2 2^4}$$

$$x_7 = \frac{6}{x_6} = \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5} = \frac{(2\cdot 4\cdot 6)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} = \frac{(3!)^2 2^6}{6!}$$

$$x_8 = \frac{7}{x_7} = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{(2\cdot 4\cdot 6)^2} = \frac{7!}{(3!)^2 2^6}$$

$$x_9 = \frac{8}{x_8} = \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7} = \frac{(2\cdot 4\cdot 6\cdot 8)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} = \frac{(4!)^2 2^8}{8!}$$

Table 1

Clearly, the closed form of x_n will depend on the parity of *n*.

Case 1: $n = 2k + 1$. Then $\boxed{x_{2k+1} = \dfrac{(k!)^2 2^{2k}}{(2k)!}}$ (6)

Case 2: $n = 2k$. Then $\boxed{x_{2k} = \dfrac{(2k-1)!}{[(k-1)!]^2 \, 2^{2k-2}}}$ (7)

The reader is reminded that two functions $f(x)$ and $g(x)$ are called *asymptotic* if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$. This does not imply that $|f(x) - g(x)|$ is bounded, as can be seen by the pair of asymptotic functions $f(x) = x^2 + x$ and $g(x) = x^2$ whose absolute difference goes to infinity. We write $f(x) \sim g(x)$ to denote that f and g are asymptotic. The reader is also reminded of Stirling's beautiful relation [1]

$$k! \sim \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \tag{8}$$

which we will use to obtain asymptotic approximations for x_{2k} and x_{2k+1}. We require the easily verified asymptotic relations (9) and (10).

$$(k!)^2 \sim \left(\frac{k}{e}\right)^{2k} 2\pi k \tag{9}$$

$$(2k)! \sim \left(\frac{2k}{e}\right)^{2k} 2\sqrt{\pi k} \tag{10}$$

Using (9) and (10) have

$$x_{2k+1} = \frac{(k!)^2 2^{2k}}{(2k)!} \sim \frac{\left(\frac{k}{e}\right)^{2k} 2\pi k 2^{2k}}{\left(\frac{2k}{e}\right)^{2k} 2\sqrt{\pi k}} = \sqrt{\pi k}$$

Obtaining an asymptotic approximation for x_{2k} will be more difficult.

$$x_{2k} = \frac{(2k-1)!}{[(k-1)!]^2 \, 2^{2k-2}} \sim \frac{\left(\frac{2k-1}{e}\right)^{2k-1} \sqrt{2\pi(2k-1)}}{\left(\frac{k-1}{e}\right)^{2k-2} 2\pi(k-1) 2^{2k-2}} \sim \frac{1}{e}\left(\frac{2k-1}{k-1}\right)^{2k-1} \frac{\sqrt{4\pi k}}{2^{2k-1}\pi} =$$

$$\frac{1}{e}\left(\frac{2k-1}{2k-2}\right)^{2k-1} \frac{2\sqrt{k}}{\sqrt{\pi}} = \frac{1}{e}\left(\frac{2k-1}{2k-2}\right)\left(1+\frac{1}{2k-2}\right)^{2k-2} \frac{2\sqrt{k}}{\sqrt{\pi}} \sim 2\sqrt{\frac{k}{\pi}} .$$

The last asymptotic approximation made use of the fact that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ .On the basis of these calculations, we have the following theorem.

Theorem 2: $\boxed{x_{2k} \sim 2\sqrt{\frac{k}{\pi}}}$ and $\boxed{x_{2k+1} \sim \sqrt{\pi k}}$ ■

Letting n = 2k in (4), we have $x_{2k}x_{2k+1} = 2k$,which is consistent with the above theorem. Furthermore, Theorem 2 implies that $\lim_{n\to\infty} x_n = \infty$. The following corollaries of Theorem 2 will be useful.

Corollary 1: $\boxed{\lim_{n\to\infty}\frac{x_{2k+1}}{x_{2k}} = \frac{\pi}{2}}$.

Proof: By Theorem 2, we have $\frac{x_{2k+1}}{x_{2k}} \sim \frac{\sqrt{\pi k}}{2\sqrt{\frac{k}{\pi}}} = \frac{\pi}{2}$.■

Corollary 2: $\boxed{\lim_{n\to\infty}\frac{x_{2k+2}}{x_{2k+1}} = \frac{2}{\pi}}$.

Proof: $x_{2k} \sim 2\sqrt{\frac{k}{\pi}}$ implies, upon replacing k by k + 1, that $x_{2k+2} \sim 2\sqrt{\frac{k+1}{\pi}}$. Then

$$\frac{x_{2k+2}}{x_{2k+1}} \sim \frac{2\sqrt{\frac{k+1}{\pi}}}{\sqrt{\pi k}} = \frac{2}{\pi}\sqrt{\frac{k+1}{k}} \sim \frac{2}{\pi}. \qquad ■$$

As a consequence of Corollaries 1 and 2, note that $\lim_{n\to\infty}\frac{x_{n+1}}{x_n}$ fails to exist. Moreover, by these corollaries, there exists a positive integer, K, such that for all $k>K$, one has $x_{2k+1}>x_{2k}$, while $x_{2k+2}<x_{2k+1}$. In other words, from some point on, terms with odd index are greater than the terms immediately before them, while the reverse is true for terms with even index. Note that this strict alternation between increasing and decreasing behavior appears to be the case for the terms in (2) with the exception of the equality of the first two terms.The next theorem requires the following Lemma which we state without proof.

Lemma 1: Let a and b be positive integers, and let p be a prime that divides b but does not divide a. Then $\frac{a}{b}$ is not an integer. ■

The reader is reminded that Bertram's Postulate [2] says that for $n > 1$, there is at least one prime p such that $n < p < 2n$.

Theorem 3: Let $n > 3$. Then x_n is not an integer.

Proof: We have two cases depending on the parity of n.

Case 1: $n = 2k + 1$, where $k > 1$. Then by (6), $x_{2k+1} = \frac{(k!)^2 2^{2k}}{(2k)!}$. By Bertram's Postulate, there exists a prime p such that $k < p < 2k$. Then p divides that denominator, but not the numerator, of x_{2k+1}. Then by Lemma 1, x_{2k+1} is not an integer.

Case 2: $n = 2k$, where $k > 1$. From Table 1, one has $x_{2k} = \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots(2k-2)}$. Then 2 divides the denominator, but not the numerator, of x_{2k}. Then by Lemma 1, $x_{2k}$is not an integer, and the theorem is proven. ■

Let $P_n = x_1x_2x_3\cdots x_n$, from which $x_{n+1}(P_n)^2 = x_1x_1x_2x_2x_3x_3\cdots x_nx_nx_{n+1}$. Recall that (4) can be written $x_nx_{n+1} = n$. Then we have $x_{n+1}(P_n)^2 = n!$ We have proven the following theorem

Theorem4: $\boxed{P_n = \sqrt{\frac{n!}{x_{n+1}}}}$ ■ (11)

The equations of Table 1, which were obtained from (4), imply that

$$x_{2k} = \frac{N_{2k}}{D_{2k}} = \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots(2k-2)} \tag{12}$$

$$x_{2k+1} = \frac{N_{2k+1}}{D_{2k+1}} = \frac{2\cdot 4\cdots(2k)}{1\cdot 3\cdots(2k-1)} \tag{13}$$

In light of (12) and (13), we turn our attention to the sequence of numerators $\{N_n\}$, whose first few terms are 1, 1, 2, 3, 8, 15, 48, 105, 384, 945. Since $x_n = \frac{N_n}{D_n}$ and $x_{n+1} = \frac{n}{x_n}$, we have $\frac{N_{n+1}}{D_{n+1}} = \frac{nD_n}{N_n}$. As we are not reducing these fractions, it follows that

$N_{n+1} = nD_n$ and $D_{n+1} = N_n$ (13)

Note that the terms 8, 15, 48, 105 and 384 of the sequence $\{N_n\}$ can be rewritten $1(3^2 - 1)$, $1(4^2 - 1)$, $2(5^2 - 1)$, $3(6^2 - 1)$ and $8(7^2 - 1)$, where the factors before the parentheses are consecutive members of the same sequence. This is not a coincidence.

By (13), $N_{n+2} = (n + 1)D_{n+1} = (n + 1)N_n$implying that $N_{n+4} = (n + 3)N_{n+2} = (n + 3)(n + 1)N_n= [(n + 2)^2 - 1]N_n$. We have proven the following theorem.

Theorem 5: $N_{n+4} = [(n + 2)^2 - 1]N_n$. ■

By the second equation in (13), a similar identity holds for the sequence, $\{D_n\}$, of denominators.

We mow turn our attention to the sequence $\sum_{n=1}^{\infty} \frac{1}{x_n}$. Replacing n by $n - 1$ in (1) yields $x_{n+1} = x_{n-1} + \frac{1}{x_n}$ in which case $\frac{1}{x_n} = x_{n+1} - x_{n-1}$. Define $x_0 = 0$,thereby enabling us to sumboth sides of this last equation from $n = 1$ to $n = m$. Note that $\frac{1}{x_1} = x_2 = 1$. We obtain $\sum_{n=1}^{m} \frac{1}{x_n} = \sum_{n=1}^{m} (x_{n+1} - x_{n-1})$. The sum on the right is $(x_2 - 0) + (x_3 - x_1) + (x_4 - x_2) + (x_5 - x_3) + \ldots + (x_{m-1} - x_{m-3}) + (x_m - x_{m-2}) + (x_{m+1} - x_{m-1}) = x_m + x_{m+1} - 1$.
Thus we have proven the following theorem.

Theorem 6: $\sum_{n=1}^{m} \frac{1}{x_n} = x_m + x_{m+1} - 1$. ■

Corollary 3: $\sum_{n=1}^{\infty} \frac{1}{x_n}$ diverges.

Proof: Let m go to infinity in Theorem 6. Then use $\lim_{n\to\infty} x_n = \infty$ which was established using Theorem 2.
■

III. A related sequence

Let a new sequence $\{y_n\}$ be defined by the seed$y_1 = y_2 = 1$, and the recursive relation

$$y_{n+2} = y_{n+1} + \frac{1}{y_n} \quad (14)$$

The first few terms are $1,1,2,3,\frac{7}{2},\frac{23}{6},\frac{173}{42}$.

Question 1: Find a recursive relation of the form $y_{n+1} = g(y_n, n)$ as was the case for $\{x_n\}$ in section I, that is, a relation such as (4).

By (14), one sees that $\{y_n\}$ is strictly increasing, unlike the sequence $\{x_n\}$. As a consequence, we have the following theorem.

Theorem 7: $\lim_{n\to\infty} y_n = \infty$.

Proof: Since $\{y_n\}$ is strictly increasing, it must either approach infinity or have a finite limit, say L.We show the latter is impossible by contradiction. Assume that $\lim_{n\to\infty} y_n = L < \infty$. Then taking the limit of both sides of (14) as n goes to infinity yields the absurd equation $L = L + \frac{1}{L}$, and we are done. ■

The next theorem yields a result similar to Theorem 6.

Theorem 8: $\sum_{n=1}^{m} \frac{1}{y_n} = y_{m+2} - 1$.

Proof: By (14), $\frac{1}{y_n} = y_{n+2} - y_{n+1}$. Then $\sum_{n=1}^{m} \frac{1}{y_n} = \sum_{n=1}^{m} (y_{n+2} - y_{n+1})$ which telescopes down to $y_{m+2} - 1$. ■

Corollary 4: $\sum_{n=1}^{\infty} \frac{1}{y_n}$ diverges. ■

IV. Another sequence whose recursive definition can be simplified

Let the sequence $\{x_n\}$ be defined by the seed $x_1 = x_2 = 1$, and the recursive relation

$$x_{n+2} = x_{n+1} + 2x_n \qquad (15)$$

which is a variant on the recursive relation of the Fibonacci sequence. The first few terms are

$$\{1, 1, 3, 5, 11, 21, 43, 85, \ldots\} \qquad (16)$$

We have the following interesting theorem which, by the way, is the reason the above sequence is included in this paper.

Theorem 9: The sequence defined by (15) and the seed $x_1 = x_2 = 1$ satisfies the recursion $x_{n+1} = 2x_n + (-1)^n$.

Proof: We use a variant of induction. The theorem is clearly true for n = 1 and n = 2. Now assume it is true for $n = k$ and $n = k + 1$. Then we have $x_{k+1} = 2x_k + (-1)^k$ and $x_{k+2} = 2x_{k+1} + (-1)^{k+1}$. Using (15) and the preceding two equations, we have

$$x_{k+3} = x_{k+2} + 2x_{k+1} = 2x_{k+1} + (-1)^{k+1} + 2[2x_k + (-1)^k] =$$

$$2(x_{k+1} + 2x_k) + [(-1)^{k+1} + 2(-1)^k] = 2x_{k+2} + (-1)^k[-1+2] =$$

$$2x_{k+2} + (-1)^k = 2x_{k+2} + (-1)^{k+2}. \quad \blacksquare$$

References

[1]. A.Taylor and W.Mann, *Advanced Calculus*. 3rd ed., Wiley, NYC, 1983.
[2]. M. Lewinter and J. Meyer, *Elementary Number Theory with Programming*,Wiley, NYC, 2015.

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