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Randomly Paintable Graphs

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Abstract
A nontrivial graph G of order n is paintable if its vertex set can be labeled with the natural numbers {1, 2, …, n} such that for eachi = 1, 2, …, n – 1, the edge (i)(i + 1)$\notin$ E(G). If anyvalid assignment of the vertex labels can be accomplished randomly,that is, at each stage of the process, one randomly selects any "available" vertex), the graph is called randomly paintable. We present several theorems and pose open questions.
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I. Introduction

We use the notation and terminology of [1]. A nontrivial graph G of order n is *paintable* [2] if its vertex set can be labeled with the natural numbers {1, 2, …, n} such that for each$i$ = 1, 2, …, n – 1, the edge (i)(i + 1)$\notin$ E(G). Infinite classes include P_n with $n \geq 4$, $C_n$with $n \geq 5$, and $K_n \cup K_m$, where $|m - n| \leq 1$ (See [3]).A graph which is not paintable is called *unpaintable*.Infinite classes of unpaintable graphs includeK_n, $K_{m,n}$, and graphs with unit radiussuch as stars.

The term 'painting' indicates that one can paint, say, houses whose locations are depicted by a graph in n days by painting a single house each day – with the condition that the fumes of a freshy painted house make it undesirable to paint an adjacent house on the following day. Another application involves scheduling a series of conferences that avoids having two consecutive conferences in the same geographic region. [2]

If anyvalid assignment of the vertex labels can be accomplished randomly, that is, at each stage of the process, one randomly selects any "available" vertex), the graph is called *randomly paintable*.The cycle, C_5, is randomly paintable.Since C_5 is vertex transitive, any vertex may be assigned the label 1. This leaves two eligible vertices for the label 2, and by symmetry considerations, it does not matter which we chose. The remainder of the labeling is forced.A glance at Figure 1 shows that the pathP_4, is paintable, but is not randomly paintable.

Figure 1:P_4 is paintable but is not randomly paintable.

In [2], it is shown that a graph is paintable if and only if its complement is *traceable*, that is, it has a spanning path. As a consequence, we have the following lemma.

Lemma 1:A graph is randomly paintable if and only if its complement is *randomly traceable*, that is, each vertex of the spanning path can be chosen randomly from among the neighbors of the previous vertex. □

II. Randomly Paintable Graphs

We will use Lemma 1 to produce an infinite class of randomly paintable graphs. We denote the complement of G by G^c.

Theorem 1:$(C_n)^c$is randomly paintable.

Proof: This follows from Lemma 1 since C_n is randomly traceable. □

Theorem 2 presents another infinite class of randomly paintable graphs.

Theorem 2:$K_n \cup K_m$is randomly paintable if and only if $m = n$.

Proof: If $m = n$, randomlyassign the odd labels of the painting in one copy of K_n and assign the even labels in the other copy. On the other hand, if $m>n$, let any vertex in $K_n$be labeled 1. Clearly the painting cannot be completed. □

Using Lemma 2, below, we will produce infinite classes of paintable graphs that are not randomly paintable.

Lemma 2: Let G be a paintable graph of order n containing two adjacent vertices, a and b, such thatG\{a, b}is paintable. Then G is not randomly paintable.

Proof: Paint G \{a, b}. To extend this painting to G, a and $b$must be assigned labels$n$ and$n$ – 1, which is impossible.□

Corollary 1: P_n is not randomly paintable.

Proof: The corollary is obvious for $n \leq 4$. For the case $n = 5$, we invoke Lemma 2, letting a and b be adjacent vertices of degree 2. For $n \geq 6$, let a and b be adjacent vertices such that one of a and b has degree 1.□

By similar reasoning, it follows that the only randomly paintable cycle is C_5.

Remark: Let G be a paintable graph containing an unpaintable subgraph H, such that G\H is paintable. Then G is not randomly paintable. This generalizes Lemma 2.

Let G be an unpaintable graph of order n. If G contains a vertex v, such that G \{v} is paintable, we call G *almost paintable*. The only almost paintable trees are K_2, P_3, and S_n.

Theorem 3: No nontrivial tree is randomly paintable.

Proof: Since no tree on two or three vertices is randomly paintable, we only consider trees on four or more vertices. Furthermore, since stars are unpaintable, we only consider trees with radius at least 2. Let T be a tree on $n \geq 4$ vertices such that rad(T) ≥ 2. Let a belong to the center of T, in which case there must exist a vertex $c$at distance 2 from a. Let b be the unique vertex adjacent to both a and c, implying that b is not an endvertex. Claim: a is not an end vertex of T. If it were, it would follow that the eccentricity, $e(b) = e(a) - 1$, which is a contradiction since a is a center vertex of T.In summation,a and b are adjacent, and neither are endvertices.

Let S = T\{a, b}. We will show that S is paintable. S is a forest with at least two components. Each component is either paintable or almost paintable. Paint each component sequentially after which each component may have at most one unlabeled vertex. Starting with any component other than the last component of the sequence, complete the painting. By Lemma 2, T is not randomly paintable. □

For $m \geq 2$, the union of m copies of $K_n$is denoted by mK_n. Note that mK_n is paintable, and by Theorem 2, $2K_n$ is randomly paintable.

Lemma 3: For $m \geq 3$, mK_n is not randomly paintable.

Proof: Paint the subgraph $(m - 1)K_n$. Since the remaining K_n is unpaintable, the lemma is an immediate consequence of the remark that generalizes Lemma 2. □

Corollary 2: The complete m-partite graph $K_{n,n,\ldots,n}$ is not randomly *traceable*. □

We will need the following lemma proven in [3]. We supply an easier proof.

Lemma 4:$K_n \times K_2$ is paintable if and only if $n \geq 3$.

Proof: There are two cases, depending on the parity of n.

Case 1:$n$is odd.Label the vertices of one copy of K_n with the numbers 1 through n. Then label the vertices of the other copy using 1′, 2′,3′, etc. such that for each k between 1 and n, k and k' are corresponding vertices. A painting can be achieved using the sequence: 1, 2′, 3, 4′, …,n, 1′, 2, 3′, 4, …, n'.

Case 2:$n$is even.Use the sequence: 1, 2′, 3, 4′, …, n', 2, 3′, 4, …, n, 1′□

It is easy to see that $K_3 \times K_2$ is randomly paintable. We will now show that for $n \geq 4$, $K_n \times K_2$ is not randomly paintable.

Theorem 4: For $n \geq 4$, $K_n \times K_2$ is not randomly paintable.

Proof: Let a be any vertex in one of the copies of K_n and let b be its corresponding vertex in the other copy. Then $(K_n \times K_2) \setminus \{a, b\} = K_{n-1} \times K_2$, which is paintable by Lemma 4. Then the theorem follows at once by Lemma 2. □

III. Miscellaneous Results

The following theorem is a convenient tool for showing that a graph is paintable. We omit the proof.
Theorem: Suppose

1) G has a paintable subgraph H
2) G \ H is paintable
3) There exists a terminal vertex,h, for a painting of H, and there exists an initial vertex, g, of a painting of G \ H, such that $gh \notin E(G)$.

Then G is paintable.

Example: Let $H = C_n$, $n \geq 5$, and create G by giving each vertex of H a pendant vertex. Then $G \setminus H = nK_1$ which is paintable. Furthermore, H is paintable. It follows that G is paintable.

The authors pose the followingquestions:

1. For various one-parameter infinite families of graphs, how many paintings are there as a function of the parameter? For example, how many paintings does P_n have?

2. What is the probability that a random graph is paintable?

References

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