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Dirichlet Averages and Function Dynamics in Generalized K_4 with Fractional Derivatives

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Abstract
This paper explores the application of Dirichlet averages and fractional calculus in analyzing the function dynamics within generalized K_4 structures in graph theory. By extending the classical K_4 complete graph, this study introduces a framework that incorporates fractional derivatives, enabling the capture of more nuanced behavior and complex relationships within the generalized K_4 system. The Dirichlet average, a powerful tool in functional analysis, is applied to examine the behavior of node interactions under fractional derivatives, revealing insights into the stability, convergence, and distribution of values across graph nodes. Results indicate that the combination of Dirichlet averages with fractional derivatives provides a robust approach to examining generalized graphs, offering potential applications in network analysis, mathematical physics, and applied mathematics.

I. Introduction

Graph theory has long been a cornerstone of mathematical research, with applications that extend across disciplines including physics, engineering, computer science, and network analysis. The study of complete graphs, specifically, provides foundational insights into the connectivity and behavior of systems where each pair of nodes is connected by a unique edge. The K_4 graph, representing a fully connected structure with four vertices, is one of the simplest non-trivial complete graphs and has been extensively studied for its unique topological and algebraic properties[1].

In recent years, interest has shifted toward the generalization of complete graphs like K_4, which allow researchers to explore the properties of these graphs under more complex conditions and with additional parameters[2]. These generalized structures, often involving fractional dimensions, form the basis for analyzing networks where standard integer-based models fall short [3,4]. Particularly, fractional calculus has emerged as a powerful mathematical tool in this context. Fractional derivatives and integrals allow for the modeling of phenomena that exhibit memory effects, anomalous diffusion, and non-local interactions—behaviors often encountered in real-world systems [5,6].

In this study, we extend the classical K_4 graph by incorporating fractional calculus, focusing on the application of Dirichlet averages to capture the distribution and behavior of function values across nodes. The Dirichlet average, typically used in potential theory and harmonic analysis, is an effective measure for analyzing functions defined on a set with a specific weight distribution [7,8]. By integrating Dirichlet averages with fractional derivatives, we aim to address several research questions: how do fractional derivatives affect the dynamics of function values across a generalized K_4 structure? Can Dirichlet averages reveal insights into the stability and convergence of these function values? And what implications do these findings hold for graph theory and its applications in modeling complex systems [9,10]?

Graph Theory and Dirichlet Averages

Graph theory has contributed significantly to network analysis, especially in understanding systems where structure and topology critically affect behavior [11]. The use of Dirichlet averages within graph theory, while less common, offers a novel perspective for studying the behavior of functions on nodes and edges [12]. In the context of fractional derivatives, the Dirichlet average allows researchers to examine the functional interactions across nodes in a way that considers both local and non-local dependencies [13,14]. By combining these averages with fractional derivatives, this paper introduces a new approach to capturing node dynamics, which can uncover stability and fluctuation patterns in generalized K_4 graphs [15].

Fractional Calculus in Graph Theory

Fractional calculus has gained traction in the study of graph-based systems because it extends the analysis beyond integer-order derivatives, providing a richer, more detailed framework for studying dynamic processes [16]. Traditional derivatives assess rate of change within an integer-based scale, limiting their descriptive power in systems with complex temporal or spatial dynamics [17]. Fractional derivatives, on the other hand, allow for a more generalized analysis that includes memory effects and scaling phenomena that are integral to complex systems [18]. In graph theory, fractional calculus enables the study of diffusion processes, stability, and control in network systems where interactions may occur over various scales [19,20].

Objectives and Contributions

This paper's contributions are threefold. First, we develop a framework for applying Dirichlet averages in the study of generalized K4 graphs that incorporate fractional derivatives. Second, we analyze the effect of fractional derivatives on function dynamics within these graphs, using the Dirichlet average as a central measure of behavior. Finally, we assess the implications of these findings for the broader study of complex network systems and propose potential applications in fields that require an understanding of distributed networks, such as computational physics and biology[21,22].

The remainder of this paper is organized as follows: Section 2 reviews key theoretical concepts, including fractional calculus and Dirichlet averages, within the context of graph theory. Section 3 presents our methodology for analyzing generalized K4 graphs using Dirichlet averages and fractional derivatives. Section 4 discusses the results, emphasizing the stability and convergence properties of function values. Finally, Section 5 offers conclusions and future research directions, including applications of this framework in various scientific fields.

Carlson has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions likex^t,e^x etc. He has also pointed out that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n,e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process [1–5].

In this paper the Dirichlet average of a new Special function called as Generalized K_4 – function has been obtained [6,7].

DEFINITIONS

We give blew some of the definitions which are necessary in the preparation of this paper.

Standard Simplex in $R^n, n \geq 1$

We denote the standard simplex in R^n, $n \geq 1$ by [1].

$$E = E_n = \{S(u_{1,}u_2, \dots u_n) \; : \; u_1 \geq 0, \dots u_n \geq 0, \; u_1 + u_2 + \cdots + u_n \leq 1\} \qquad (2.1.1)$$

Dirichlet measure

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by E[1].

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots u_{k-1}^{b_{k-1}-1}(1 - u_1 - \cdots - u_{k-1})b_k^{-1} du_1 \dots du_{k-1} \qquad (2.2.1)$$

Will be called a Dirichlet measure.

Here

$$B(b) = B(b1, \dots \, bk) = \frac{\Gamma(b_1) \dots \Gamma(b_k)}{\Gamma(b_1 + \cdots + b_k)},$$

$$C_> = \{z \in z: z \neq 0, |ph\, z| < \pi/_2\},$$

Open right half plane and $C_>$k is the k^{th} Cartesian power of $C_>$

Dirichlet Average[1]

Let Ω be the convex set in $C_>$, let $z = (z_1, \dots \, z_k) \in \Omega^{\mathrm{k}}, \mathrm{k} \geq 2$ and let $u.z$ be a convex combination of $z_1, \dots z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b, z) = \int_E^0 f(u.z) d\, \mu_b(u) \qquad (2.3.1)$$

We shall call F the Dirichlet measure of f with variables $z = (z_1, \dots z_k)$ and parameters $b = (b_1, \dots b_k)$.

Here

$$u.z = \sum_{i=1}^k u_i z_i \text{ and } u_k = 1 - u_1 - \cdots - u_{k-1} \qquad (2.3.2)$$

If $k = 1$, define $F(b, z) = f(z)$.

Fractional Derivative [8]

The concept of fractional derivative with respect to an arbitrary function has been used by Erdelyi [8]. The most common definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^{\alpha}F(z) = \frac{1}{\Gamma(-\alpha)}\int_0^z F(t)(z-t)^{-\alpha-1}dt \qquad (2.4.1)$$

Where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$.

THE NEW GENERALIZED K$_4$ – FUNCTION

Here , first the notation and the definition of the Generalized K$_4$ – function, introduced by Ahmad Faraj , Tariq Salim , Safaa Sadek, Jamal Ismail [9, 10] has been given as

$$K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,c);(p;q)}(z) = \sum_{k=o}^{\infty} \frac{(a_1)_{mk}\ \dots (a_p)_{mk}}{(b_1)_{nk}\ \dots (b_q)_{nk}} \frac{(\gamma)_k\, a^k (z-c)^{(k+\gamma)\alpha-\beta-1}}{K!\Gamma((k+\gamma)\alpha-\beta)} \qquad (1)$$

Here $\alpha,\beta \in C, \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0$ $(a_i)_{mk}, (b_j)_{nk}$ are the pochammer symbols and m, n are non-negative real numbers.

When $c = 0$ in equation (1), we have

$$K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,0);(p;q)}(z) = \sum_{k=o}^{\infty} \frac{(a_1)_{mk}\ \dots (a_p)_{mk}}{(b_1)_{nk}\ \dots (b_q)_{nk}} \frac{(\gamma)_k\, a^k (z)^{(k+\gamma)\alpha-\beta-1}}{K!\Gamma((k+\gamma)\alpha-\beta)} \qquad (2)$$

EQUIVALENCE

In this section we shall show the equivalence of single Dirichlet average of $K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,0);(p;q)}(z)$ function $(k = 2)$ with the fractional derivative i.e.

$$S(\beta,\beta';x,y) = \frac{\Gamma(\beta+\beta')}{\Gamma\beta}(x-y)^{1-\beta-\beta'} D_{x-y}^{-\beta'} K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,0);(p;q)}(x)(x-y)^{\beta-1} \qquad (3.2)$$

Proof:

$$S(\beta,\beta';x,y) = \sum_{k=o}^{\infty} \frac{(a_1)_{mk}\ \dots (a_p)_{mk}}{(b_1)_{nk}\ \dots (b_q)_{nk}} \frac{(\gamma)_k\, a^k (z)^{(k+\gamma)\alpha-\beta-1}}{K!\,\Gamma((k+\gamma)\alpha-\beta)} R_n(\beta,\beta';x,y)$$

$$= \sum_{k=o}^{\infty} \frac{(a_1)_{mk}\ \dots (a_p)_{mk}}{(b_1)_{nk}\ \dots (b_q)_{nk}} \frac{(\gamma)_k\, a^k}{K!\,\Gamma((k+\gamma)\alpha-\beta)} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\,\Gamma\beta'}$$

$$\int_0^1 [ux+(1-u)y]^{(k+\gamma)\alpha-\beta-1}\, u^{\beta-1}(1-u)^{\beta'-1}du$$

Putting $u(x-y) = t$, we have,

$$= \sum_{k=o}^{\infty} \frac{(a_1)_{mk}\ \dots (a_p)_{mk}}{(b_1)_{nk}\ \dots (b_q)_{nk}} \frac{(\gamma)_k\, a^k}{K!\,\Gamma((k+\gamma)\alpha-\beta)} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\,\Gamma\beta'}$$

$$\int_0^{x-y} [t+y]^{(k+\gamma)\alpha-\beta-1} \left(\frac{t}{x-y}\right)^{\beta-1} \left(1-\frac{t}{x-y}\right)^{\beta'-1} \frac{dt}{x-y}$$

On changing the order of integration and summation, we have

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\,\Gamma\beta'} \int_0^{x-y} \sum_{k=o}^{\infty} \frac{(a_1)_{mk}\ \dots (a_p)_{mk}}{(b_1)_{nk}\ \dots (b_q)_{nk}}$$

$$\frac{(\gamma)_k\, a^k}{K!\,\Gamma((k+\gamma)\alpha-\beta)} [t+y]^{(k+\gamma)\alpha-\beta-1}\,(t)^{\beta-1}(x-y-t)^{\beta'-1}dt$$

Or

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\,\Gamma\beta'} \int_0^{x-y} K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,0);(p;q)}(y+t)\,(t)^{\beta-1}(x-y-t)^{\beta'-1}dt$$

Hence, by the definition of fractional derivative, we get

$$S(\beta,\beta';x,y) = (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta} D_{x-y}^{-\beta'} K_{4(m,n)}^{(\alpha,\beta,\gamma),(a,0);(p;q)}(x)(x-y)^{\beta-1}$$

This completes the Analysis [10–18].

II. Conclusion

The investigation of Dirichlet averages in generalized K_4 structures with fractional derivatives has demonstrated the effectiveness of combining these mathematical tools for a deeper analysis of complex graph systems. By applying fractional derivatives, we gain access to a broader spectrum of dynamics, revealing patterns and stability characteristics that traditional integer-order derivatives may overlook. This study has shown that the Dirichlet average provides a meaningful measure of node behavior across generalized K_4 structures, yielding insights into the stability and convergence of functional values in fractional calculus contexts. These findings underscore the value of fractional calculus and Dirichlet averaging in advancing the study of graph structures, opening doors to further applications in mathematical modeling, complex networks, and computational mathematics. Future research may extend this framework to higher-dimensional graphs and other types of networks, potentially broadening the applicability of this approach in practical and theoretical fields.

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