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Lyapunov-Type Inequalities For *N*th Order Forced Differential Equations With Mixed Nonlinearities

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Abstract
In this paper, some counter-examples are given to show that the results of [1] are not correct. Moreover, the reasons for these mistakes are given and some possible improvements are suggested.
Keywords: *Lyapunov-type inequality; nth order equations; mixed nonlineariries*

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I. Introduction

The well-known Lyapunov inequality [4] for the following second-order linear differential equation:

$x''(t) + q(t)x(t) = 0$ (1)

where $q(t) \in L^1[a, b]$ is a real valued function, states that if $a < b$ are consecutive zeros of a nonzero solution $x(t)$ of (1), then the following inequality and the constant 4 is sharp, which means that it can not be replaced by larger ones.

Later Wintner [5] replaced the function $|q(t)|$ in (2) by the function $q^+(t)$, where $q^+(t) = \max\{q(t), 0\}$.

That is, he obtained the following inequality:

In [3], by using the Green function method, Hartmann obtained an inequality which is more sharper than both (2) and (3):

Since for all $t \in (a, b)$, $(b - t)(t - a) \leq \frac{(b-a)^2}{4}$, inequality (4) implies (2) and (3).

The Lyapunov type inequalities have been applied to oscillation and Sturmian Theory, disconjugacy, eigenvalue problems and many other properties of the solutions. These inequalities and their generalizations to higher order differential equations, to Hamiltonian systems, nonlinear differential equations, See, for example, [1-5] and the references cited therein.

In paper [1], the authors considered the following nth order forced differential equation of the form No sign restriction is improsed on the forcing term and the nonlinearities satisfying

$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_j < 1 < \cdots < \alpha_m < 2.$

The authors of [1] obtained the following main results:

Theorem 2.2 (Lyapunov type inequality) Let $x(t)$ be a nontrivial solution of (5) satisfying boundary conditions (6). If $x(t) \neq 0$ in (a_j, a_{j+1}), $j = 1, 2, \cdots, r - 1$, then the inequality

Theorem 2.3 (Lyapunov type inequality) Let $x(t)$ be a nontrivial solution of (5) satisfying the following $(k, n - k)$ conjugate boundary conditions holds, where the functions $Q_m(t)$ and $Q_m(t)$ are defined in (10) and $\Phi(t) = \max_{t\in[a,b]} \{ k(t-a)^{n-k}(t+b-2a)^{k-1}, (n-k)(t-a)^k(2b-a-t)^{n-k-1} \}$ for $k = 1, 2, \cdots, n - 1$.

Consider the following boundary value problem:

Theorem 3.4 (Hartman type inequality) Let $x(t)$ be a nontrivial solution of (11) satisfying the following $(k, n - k)$ conjugate boundary conditions holds, where the functions $Q_m(t)$ and $Q_m(t)$ are defined in (8)

Counter-Examples

Let us consider equation (5) with $a_1 = 0 < a_2 = \pi$, $0 < \alpha_1 < 1 < \alpha_2 < 2$.

but the right side of (7) is $\frac{9 \cdot 4^5}{\pi^8}$, which is a positive constant independent of the variable of g.

When we let $g \to 0^{+}$, then the left side of (7) tends to 0, but the right side of (7) remains a positive constant. This yields a contraction! Therefore the result of Theorem 2.2 is incorrect.

Counter-example 2. If we let $a = 0 < b = 1$, $x(t) = gt^{3}(1-t)^{2}$, where $g \in (0, 1)$ is a positive constant. Then $x(0) = x'(0) = x''(0) = 0$, $x(1) = x'(1) = 0$, $x^{(5)}(t) = 5! = 120$, $k_1 = 2$, $k_2 = 1$, $r = 2$, $n = k_1 + k_2 + r = 2 + 1 + 2 = 5$.

Let

$$q_1(t) = q_2(t) = g,\ f(t) = 5!g + g\ g^{\alpha_1} t^{3\alpha_1} (1-t)^{2\alpha_1} + g^{\alpha_2} t^{3\alpha_2} (1-t)^{2\alpha_2} .$$

Then it is easy to verify that $x(t)$ is a nonzero solution of (5) and (10).

Moreover, $x(t) > 0$, $t \in (0, 1)$, $\max_{0\le t\le 1} x(t) = x(\,^{3}\,) = \underline{^{36}}\, g > 0$. It is easy to verify that the left side of (44)

When we let $g \to 0^{+}$, the left side of (44) tends to 0^{+}, but the right side of (44) remains a positive constant, this is a contraction! Therefore the result of Theorem 2.3 is incorrect.

Counter-example 3. If we define $a = 0 < b = 1$, $x(t) = gt^{3}(1-t)^{2}$, $q_1(t) = q_2(t) = g$ and $f(t)$ is the same as Counter-example 2, the it is easy the verify that the result of Theorem 3.4 is incorrect. From above counter-examples, we see that the results of Theorem 2.2 and Theorem 2.3 and Theorem 3.4 are incorrect.

The authors claims that inequality (15) is only possible when $R_1R_2 > \frac{1}{4}$. That is, the condition $4R_1R_2 > 1$ is a necessary condition for the existence of a nontrivial solution of (7). But after we carefully checking this claim, it seems that the authors made a mistake. In fact, for $R_1 > 0$, $R_1x^{2}(c) - x(c) + B > 0$ is equivalent to This inequality dose not implies that $4R_1R_2 > 1$ since the equation (7) is not a linear equation of type (1) or (5). In (1) and (5), if $x(t)$ is a nonzero solution, then for any real number λ, $\lambda x(t)$ is also a solution of (1) or (5), therefore, the value $x(c) = \max_{a\le t\le b} x(t)$ when $x(t) > 0$, $t \in (a, b)$, can take any positive value if we choose suitable $\lambda > 0$. In this case, let $x(c) = \frac{1}{2R_1}$, then (16) implies that $4R_1R_2 > 1$. But for nonlinear equation (7), the value $x(c)$ or $|x(c)|$ may belong to some interval $I = [x_0, x_1]$, which may be a single point or which does not contain the value $\frac{1}{2R_1}$, therefore, one can not conclude that (16) implies $4R_1R_2 > 1$. Since all the proofs of theorems in [1] relay on this claim, the results of [1] are therefore all incorrect.

Besides, Lemma 2.1 in [1] states the if A is positive and B, z are nonnegative, then for any $\alpha \in (0, 2)$ with equality holding if and only if $B = z = 0$. In fact, the claim that the equality holds if and only if $B = z = 0$ is incorrect. In fact, let $A = B = \alpha = 1$, then inequality (17) reduces to it is ease to see that the equality holds if and only if $z = \frac{1}{2}$. It follows from this conclusion that the result of Lemma 2.1 is incorrect also.

Besides, the results of [2] are incorrect for the same reason.

II. Some Corrections To The Results In [1]

1. The authors of [1] should add some sufficient conditions to Theorem 2.2 and Theorem 2.3 and Theorem 3.4, for example, they may assume that all solutions of (7) can take any real value. In this case, the inequality

(16) could be satisfied if $x(c)$ can take any real value. In this case, the inequality (16) implies that $4R_1R_2 > 1$.

Under this assumption, the results of [1] and [2] remain correct.

III. Data Availability

No data was used for the research described in the article.

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