IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 21, Issue 4, Ser. 1 (Jul. – Aug. 2025), PP 01-04
www.iosrjournals.org

Null Vertex in Graphs

Dr. Sunil M.P
Department of Mathematics, Government College, Kottayam, Kerala, India. 686013.

Abstract: *In graphs, circular distance between two vertices is the sum of geodesic and detour distance between them. Based on circular distance, we introduce a novel vertex, termed as null vertex, that is distinct from interior and boundary vertices and discuss its presence in some graphs.*
Keywords*: circular distance, boundary vertex, interior vertex, null vertex.*

Date of Submission: 07-07-2025 Date of Acceptance: 17-07-2025

I. INTRODUCTION AND PRELIMINARIES

Graph theory particularly deals with network of points connected by lines. Boundary vertices and interior vertices, which are integral parts of a graph are discussed in [1],[2]. Various authors have studied about different types of distance concepts in graphs. In a graph, circular distance (C-distance) between two vertices is the sum of geodesic and detour distance between them [6]. The extensive expansion of networks in all spheres of life has made the study of distance concept an important one, particularly in graph theory. Null vertex in fuzzy graphs and bipolar fuzzy graphs are discussed in [4],[5].

In graphs, when we consider geodesic distance as the distance between two vertices, all vertices are either boundary vertices or both interior and boundary vertices. But, when C-distance is used instead of geodesic distance, we identify that there may exist some vertices, null vertices, which are neither boundary vertices nor interior vertices. We make a study on the presence of null vertices based on C-distance in wheel graphs, helm graphs and gear graphs.

Definition 1.1. [3] Consider two vertices x and y in a connected graph G. The geodesic distance $d(x, y)$ is the length of the shortest $x - y$ path and the detour distance $D(x, y)$ is the length of the longest $x - y$ path.

Definition 1.2. [6] Circular distance (C-distance) $d_c(x, y)$ between x, y in G is

$$d_c(x, y) = \begin{cases} d(x, y) + D(x, y), & x \neq y \\ 0, & x = y \end{cases}$$

C-distance is a metric on the set of all vertices of G.

II. MAIN RESULTS

Definition 2.1. A vertex y in a connected graph G is a boundary vertex of a vertex x in G with respect to C-distance if $d_c(x, y) \geq d_c(x, z)$, for each neighbour z of y and if y is a boundary vertex of some vertex of G, then y is a boundary vertex of G. A vertex z in G is an interior vertex of G with respect to C-distance if for each vertex x in G, there exists a vertex y in $G, x \neq z \neq y$ with $d_c(x, y) = d_c(x, z) + d_c(z, y)$.

Definition 2.2. In a connected graph G, a vertex y is a null vertex with respect to C-distance if y is neither an interior vertex nor a boundary vertex.

Lemma 2.3. In a connected graph, vertex with degree 1 is a boundary vertex.

Proof: Consider the vertices $x_1, x_2, \ldots\ldots, x_n$ in G with deg $(x_1) = 1$. Let x_2 be the one and only one neighbour of x_1. So, $d_c(x_i, x_1) \geq d_c(x_i, x_2), i \neq 1$. i.e., x_1 is a boundary vertex.

Theorem 2.4. A boundary vertex in a graph is not an interior vertex with respect to C-distance.

Proof: Let y be a boundary vertex of the vertex x with respect to C-distance. Then, $d_c(x,y) \geq d_c(x,u)$, for each neighbour u of y. But, if y is an interior vertex, $\exists$ a vertex z, $x \neq y \neq z$, where y lies between x and z. Let $P: x = y_1, y_2, \dots y = y_k, y_{k+1}, \dots y_m = z$ be a $x - z$ path, $1 < k < m$. Then $y_{k+1} \in N(y)$, the neighbourhood of y. So, $d_c(x, y_{k+1}) > d_c(x,y)$, a contradiction. Thus, y is not an interior vertex with respect to C-distance.

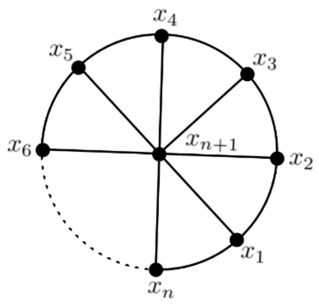

Figure1. Wheel Graph W_n

Theorem 2.5. There exists a unique null vertex with respect to C-distance for the Wheel graph $W_n, n \geq 4$.

Proof: Consider $W_n, n \geq 4$ in figure 1 with vertices $x_i, 1 \leq i \leq n+1$, where x_{n+1} is the apex vertex joining $x_1, x_2, \dots\dots x_n$, $\deg(x_i) = 3$, $1 \leq i \leq n$, $\deg(x_{n+1}) = n$.

$$d_c(x_i, x_j) = \begin{cases} n+1, when\ \ d(x_i, x_j) = 1 \\ n+2, when\ \ d(x_i, x_j) = 2 \end{cases}$$

Consider the vertex x_1. The neighbours of x_1 are x_j where, $d(x_1, x_j) = 1$. $d_c(x_{n+1}, x_1) = n+1$. x_1 is a boundary vertex of x_{n+1} with respect to C-distance since, $d_c(x_{n+1}, x_1) \geq d_c(x_{n+1}, x_j)$, for all neighbours x_j of x_1. Similarly, $x_2, x_3, \dots\dots x_n$ are also boundary vertices of x_{n+1} with respect to C-distance.

Neighbours of x_{n+1} are $x_i, 1 \leq i \leq n$. We have, $d_c(x_i, x_{n+1}) = n+1$ and $d_c(x_i, x_j) = n+2$ when $d(x_i, x_j) = 2$. $x_i, 1 \leq i \leq n$ does not have x_{n+1} as a boundary vertex with respect to C-distance since, $d_c(x_i, x_{n+1}) < d_c(x_i, x_j)$, for some neighbours x_j of x_{n+1} with $d(x_i, x_j) = 2$.

But, $d_c(x_i, x_{n+1}) + d_c(x_{n+1}, x_j) = 2n+2$, $i \neq j$. So, x_{n+1} is not an interior vertex of W_n with respect to C-distance since, $d_c(x_i, x_j) \neq d_c(x_i, x_{n+1}) + d_c(x_{n+1}, x_j)$, $i \neq j$. Thus, x_{n+1} is a null vertex with respect to C-distance.

Remark 2.6. In the Wheel graph W_3, $d_c(x_i, x_j) = 4$ for all $i \neq j$ and all the vertices are boundary vertices.

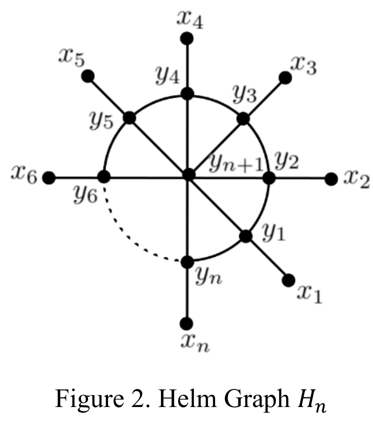

Figure 2. Helm Graph H_n

Theorem 2.7. There is a unique null vertex for the Helm graph $H_n, n \geq 4$ with respect to C-distance.

Proof: Consider $H_n, n \geq 4$ in figure 2, having vertices $x_i, y_i, i = 1,2, \dots n, deg(x_i) = 1, deg(y_i) = 4$ and the apex vertex y_{n+1}, $deg(y_{n+1}) = n$.

$$d_c(y_i, y_j) = \begin{cases} n+1, when \ d(y_i, y_j) = 1 \\ n+2, when \ d(y_i, y_j) = 2 \end{cases}$$

$$d_c(x_i, x_j) = \begin{cases} n+5, when \ d(x_i, x_j) = 3 \\ n+6, when \ d(x_i, x_j) = 4 \end{cases}$$

$$d_c(x_i, y_j) = \begin{cases} 2, \quad when \ d(x_i, y_j) = 1 \\ n+3, when \ d(x_i, y_j) = 2 \\ n+4, when \ d(x_i, y_j) = 3 \end{cases}$$

Here, $x_i, 1 \leq i \leq n$ are boundary vertices with respect to C-distance by lemma 2.3. For the vertex x_1, there exists vertices $x_i, i = 2, \dots n$ with $d_c(x_1, x_i) = d_c(x_1, y_1) + d_c(y_1, x_i)$. Also, for the vertex x_1, there exists vertices $y_j, j = 2,3, \dots n+1$ with $d_c(x_1, y_j) = d_c(x_1, y_1) + d_c(y_1, y_j)$. Thus, y_1 is an interior vertex with respect to C-distance. Similarly, $y_j, 2 \leq j \leq n$ are also interior vertices with respect to C-distance.

Consider the vertex y_{n+1}. Since, $d(x_1, y_{n+1}) = 2$, $d_c(x_1, y_{n+1}) = n+3$. The neighbours of y_{n+1} are $y_j, j = 1, 2, , \dots n$. Also, $d_c(x_1, y_j) = n+4$ when $d(x_1, y_j) = 3$. Clearly, $d_c(x_1, y_{n+1}) < d_c(x_1, y_j)$, for some neighbours y_j of y_{n+1} with $d(x_1, y_j) = 3$. So, y_{n+1} is not a boundary vertex of x_1 with respect to C-distance. Same is the case with $x_i, 2 \leq i \leq n$. Also, since $d(y_1, y_{n+1}) = 1$, $d_c(y_1, y_{n+1}) = n+1$ and $d_c(y_1, y_j) = n+2$ when $d(y_1, y_j) = 2$. i.e., y_{n+1} is not a boundary vertex of y_1 with respect to C-distance. Same is the case with $y_2, y_3, \dots \dots y_n$.

For x_i, a vertex x_j or $y_j, i \neq j$ does not exist with $d_c(x_i, x_j) = d_c(x_i, y_{n+1}) + d_c(y_{n+1}, x_j)$ or $d_c(x_i, y_j) = d_c(x_i, y_{n+1}) + d_c(y_{n+1}, y_j)$. So, y_{n+1} is not an interior vertex with respect to C-distance. Thus, y_{n+1} is a null vertex with respect to C-distance.

Remark 2.8. In the Helm graph H_3, all the vertices are boundary vertices.

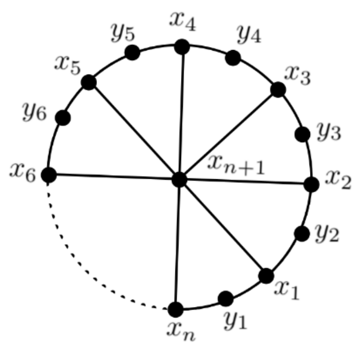

Figure 3. Gear Graph G_n

Theorem 2.9. The Gear graph $G_n, n \geq 4$ has n null vertices with respect to C-distance.

Proof: Consider G_n in figure 3 having vertices $x_i, y_i, i = 1,2, \dots n, \deg(x_i) = 3, \deg(y_i) = 2$ and the apex vertex x_{n+1}, deg $(x_{n+1}) = n$. Here, $d_c(x_i, x_j) = 2n$, $d_c(x_{n+1}, y_i) = 2n+2$.

$$d_c(x_i,y_j) = \begin{cases} 2n, & when\ \ d(x_i,y_j) = 1 \\ 2n+2, & when\ \ d(x_i,y_j) = 3 \end{cases}$$

$$d_c(y_i,y_j) = \begin{cases} 2n+2, when\ \ d(y_i,y_j) = 2 \\ 2n+4, when\ \ d(y_i,y_j) = 4 \end{cases}$$

Consider y_1. The neighbours of y_1 are $x_k, k = 1, n.$ y_1 is a boundary vertex of y_2 since $d_c(y_2,y_1) \geq d_c(y_2,x_k)$, for the neighbours x_k of y_1. Similarly, $y_2, y_3, \ldots\ldots y_n$ are boundary vertices with respect to C-distance.

Consider x_1. The neighbours of x_1 are y_1, y_2 and x_{n+1}. Since $d_c(x_j,x_1) < d_c(x_j,y_1), j = 2,3, \ldots (n-1)$ and $d_c(x_j,x_1) < d_c(x_j,y_2), j = n, n+1,$ x_1 is not a boundary vertex of x_j with respect to C-distance.

Since, $d_c(y_j,x_1) < d_c(y_j,y_1), j = 2,3, \ldots.(n-1)$ and $d_c(y_j,x_1) < d_c(y_j,y_2), j = 1, n,$ the vertex x_1 is not a boundary vertex of y_j with respect to C-distance. Similarly, $x_i, 2 \leq i \leq n$ are also not boundary vertices with respect to C-distance.

For x_1, a vertex x_j or y_j does not exist with $d_c(x_i,x_j) = d_c(x_i,x_1) + d_c(x_1,x_j)$ or $d_c(x_i,y_j) = d_c(x_i,x_1) + d_c(x_1,y_j)$. So, x_1 is not an interior vertex with respect to C-distance. Similarly, $x_i, 2 \leq i \leq n$ are also not interior vertices with respect to C-distance. Hence, $x_i, 1 \leq i \leq n$ are null vertices with respect to C-distance.

Corollary 2.10. In the Gear graph G_3, all the vertices are boundary vertices.

III. CONCLUSION

In a graph, when we consider geodesic distance as the distance between two vertices, there exists either boundary vertices or both boundary and interior vertices. We established that when circular distance is taken instead of geodesic distance, there may exists some other vertices, termed as null vertices, which are distinct from interior and boundary vertices. We proved the existence of null vertices with respect to C-distance in wheel graphs, helm graphs and gear graphs for $n \geq 4$.

REFERENCES

[1]. G. Chartrand and P. Zhang, A First Course in Graph Theory, Dover Publications, New York, 2012.

[2]. G. Chartrand, D. Erwin, G. Johns and P. Zhang, Boundary vertices in graphs, Discrete Mathematics, vol. 263, pp. 25–34, 2003. doi.org/10.1016/S0012-365X (02)00567-8

[3]. G. Chartrand, H. Escuadro and P. Zhang, Detour distance in graphs, J Combin. Math. Combin. Compute., vol. 53, pp. 75–94, 2005.

[4]. M.P. Sunil and J. Suresh Kumar, On Null Vertex in Fuzzy Graphs, International Journal of Analysis and Applications, vol. 22, 43, 2024. https://doi.org/10.28924/2291-8639-22-2024-43.

[5]. M.P. Sunil and J. Suresh Kumar, On Null Vertex in Bipolar Fuzzy Graphs, International Journal of Analysis and Applications, vol.22, 142, 2024. https://doi.org/10.28924/2291-8639-22-2024-142.

[6]. J. Veeranjaneyulu and P.L.N. Varma, Study of circular distance in graphs, Turkish Journal of Computer and Mathematics Education, vol. 12, no. 2, pp. 2437-2444, 2021. https://turcomat.org/index.php/turkbilmat/article/view/2067.