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Rough Precompactness On Topological Simple Rough Groups

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Abstract:
In this paper, we introduce a new definition of rough precompact in topological simple rough groups. We develop the fundamental properties of this concept and examine its relationship to compactness and rough complete. Further we investigate how this concept behaves under rough subgroup and product topology.
***Keyword**: Topological simple rough groups, Rough filter, Rough ultrafilter, Rough complete, Rough precompact.*

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I. Introduction

The rough set theory, initially proposed by Pawlak (1982), has been utilized as an effective mathematical tool for modeling and processing incomplete information. In recent years, rough sets have been integrated with various mathematical theories such as algebra and topology. In 1994, Biswas and Nanda introduced the concept of rough group and rough subgroups, which are based on upper approximation and are independent of lower approximation. Miao et al. have enhanced the definitions of rough group and rough subgroup, and have demonstrated their new properties. Conversely, Kuroki and Wang outlined certain properties of lower and upper approximations in relation to the normal subgroups in 1996. Bagirmaz et al. (2016) proposed the concept of topological rough groups, expanding the idea of a topological group to encompass the algebraic structures of rough groups.

Precompactness is a fundamental concept in topology, particularly in the study of topological groups, it plays a crucial role in understanding the behavior of group actions, completeness and compactness. In this paper, we introduce our new definition "rough precompact" in topological simple rough group and examine its basic properties. Also, we investigate how this notion interacts with rough complete and study its behavior under rough subgroups and product constructions.

II. Preliminaries

Definition 2.1.[3] Let $K = (U, R)$ be an approximation space and $*$ be a binary operation defined on U. A subset G of universe U is called a rough group if the following properties are satisfied:
(i) $\forall$ x, y $\in$ G, $x*y \in \overline{G}$;
(ii) Association property holds in $\overline{G}$;
(iii) $\exists$ e $\in \overline{G}$ such that $\forall$ x $\in$ G, $x*e = e*x = x$; e is called the rough identity element of rough group G;
(iv) $\forall$ x $\in$ G, $\exists$ y $\in$ G such that $x*y = y*x = e$; y is called the rough inverse element of x in G;

Definition 2.2.[8] A topological rough group is a rough group (G, $*$) together with a topology T on $\overline{G}$ satisfying the following two properties:
(i) the mapping f: G × G → $\overline{G}$ defined by f(x, y) = xy is continuous with respect to product topology on G × G and the topology T_G on G induced by T,
(ii) the inverse mapping g: G → G defined by g(x) = x^{-1} is continuous with respect to the topology T_G on G induced by T.

Definition 2.3.[10] A rough group $G_{\mathfrak{R}}$ is called a simple rough group if it contains no proper non-trivial rough normal subgroups.
That is, $G_{\mathfrak{R}}$ has only the rough normal subgroups {e} and $G_{\mathfrak{R}}$.

Definition 2.4.[10] A topological simple rough group is a simple rough group $(G_{\mathfrak{R}}, *)$ together with a topology $\bar{\tau}$ on $\overline{G_{\mathfrak{R}}}$ satisfying the following two properties:
(i) The mapping f: $G_{\mathfrak{R}} \times G_{\mathfrak{R}} \to \overline{G_{\mathfrak{R}}}$ defined by f(x, y) = xy, x, y $\in G_{\mathfrak{R}}$ is continuous with respect to the product topology on $G_{\mathfrak{R}} \times G_{\mathfrak{R}}$ and the topology τ on $G_{\mathfrak{R}}$ induced by $\bar{\tau}$
(ii) The inverse mapping g: $G_{\mathfrak{R}} \to G_{\mathfrak{R}}$ defined by g(x) = x^{-1}, x $\in G_{\mathfrak{R}}$ is continuous with respect to the topology τ on $G_{\mathfrak{R}}$ induced by $\bar{\tau}$.

Definition 2.5.[12] A Rough filter on a topological simple rough group $G_{\mathfrak{R}}$ is a family $\mathcal{F}$ of non-empty subsets of $\overline{G_{\mathfrak{R}}}$ satisfying the following conditions:
(i) if U and V are in $\mathcal{F}$, then $U \cap V \in \mathcal{F}$;
(ii) if $U \in \mathcal{F}$ and $U \subseteq V \subseteq \overline{G_{\mathfrak{R}}}$, then $V \in \mathcal{F}$.

Definition 2.6.[12] A rough filter $\mathcal{F}$ is called a rough cauchy filter on a topological simple rough group $G_{\mathfrak{R}}$ if for every open neighborhood U of an identity element e in $\overline{G_{\mathfrak{R}}}$, there exists elements $x, y \in \overline{G_{\mathfrak{R}}}$ and $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq xU$ and $F_2 \subseteq Uy$.
In other words, a rough filter $\mathcal{F}$ in a topological simple rough group $G_{\mathfrak{R}}$ is called a rough cauchy filter if for each open neighborhood U of e in $\overline{G_{\mathfrak{R}}}$, there is an element $F \in \mathcal{F}$ such that $FF^{-1} \subseteq U$.

Definition 2.7.[12] A topological simple rough group $G_{\mathfrak{R}}$ is rough complete if every rough cauchy filter on $G_{\mathfrak{R}}$ converges.

Theorem 2.8.[3] A necessary and sufficient condition for a subset H of rough group G to be a rough subgroup is that:
(i) $\forall$ x, y $\in$ H, x$*$y $\in \overline{G}$;
(ii) $\forall$ x $\in$ H, $x^{-1} \in$ H.

Proposition 2.9.[10] Let $G_{\mathfrak{R}}$ be a topological simple rough group. If U $\subseteq \overline{G_{\mathfrak{R}}}$ is an open set with e $\in$ U, then there exists a symmetric open set V of e in $G_{\mathfrak{R}}$ such that VV $\subseteq$ U.

Lemma 2.10.[11] Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$ and W be a neighbourhood of e in $\overline{G_{\mathfrak{R}}}$. Then there is an open set U of e in $G_{\mathfrak{R}}$ such that $U \subseteq U^n \subseteq W$, for every $n \in \mathbb{N} - \{0\}$.

Theorem 2.11.[12] Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. Then the product $\prod_{i \in I} G_{\mathfrak{R}}^{(i)}$ is rough complete, where $G_{\mathfrak{R}}^{(i)}$ are rough complete groups.

Theorem 2.12.[4] A topological space X is compact iff every ultrafilter on X is convergent.

Throughout this paper, we consider X be the universal set, $G_{\mathfrak{R}}$ be a simple rough group with identity e and $\overline{G_{\mathfrak{R}}}$ be the upper rough approximation of $G_{\mathfrak{R}}$. Also, the corresponding topologies are denoted by $\bar{\tau}$ for $\overline{G_{\mathfrak{R}}}$ and τ for $G_{\mathfrak{R}}$ induced from $\bar{\tau}$.

III.Rough Precompact

Definition 3.1. A subset S of a topological simple rough group $G_{\mathfrak{R}}$ is *rough precompact* in $G_{\mathfrak{R}}$ if for every identity neighborhood $V \subseteq \overline{G_{\mathfrak{R}}}$, there exists a finite set $K \subseteq \overline{G_{\mathfrak{R}}}$ such that $S \subseteq KV$ and $S \subseteq VK$.

Lemma 3.2. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. Let $P \subseteq \overline{G_{\mathfrak{R}}}$ be a rough precompact subset and $D \subseteq \overline{G_{\mathfrak{R}}}$ be a dense subset of P. Then, for every identity neighbourhood $U \subseteq \overline{G_{\mathfrak{R}}}$, there exists a finite set $A \subseteq D$ such that $P \subseteq AU$ and $P \subseteq UA$.
Proof: Let U be an identity neighbourhood in $\overline{G_{\mathfrak{R}}}$. Then there exists a symmetric identity neighborhood $V \subseteq G_{\mathfrak{R}}$, such that $VV \subseteq U$. Since $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$, V is open in $\overline{G_{\mathfrak{R}}}$ and $VV \subseteq U$. Since P is a rough precompact, there exists a finite set K in $\overline{G_{\mathfrak{R}}}$ such that
$P \subseteq KV$ and $P \subseteq VK$.
Let $x \in K$ and $P \cap xV \neq \emptyset$. Since D is a dense subset of P, $D \cap xV \neq \emptyset$. Then we can choose a point $y \in D \cap xV$. That is, $y \in D$ and $y \in xV$ which implies $y = xv$, for some $v \in V$. Now consider the finite set,
$A_1 = \{y_x : x \in K \text{ and } P \cap xV \neq \emptyset\}$.
Since K is a finite set and D is a dense in P, A_1 is a finite set contained in D. Let $b \in P \subseteq KV$. Then $b = xv$, for some $x \in K$ and $v \in V$. Also, $y_x = xv \in S$ and ${y_x}^{-1} = v^{-1}x^{-1}$. That is,

$y_x{}^{-1}x = v^{-1} \in V^{-1} = V$,
V is a symmetric neighborhood, and

$$b = xv = (y_x v^{-1})v = y_x(v^{-1}v) \in y_x VV \subseteq y_x U$$

which implies $b \in A_1U$. Therefore, $P \subseteq A_1U$. Similarly, we can prove $P \subseteq UA_2$, where
$A_2 = \{z_x : x \in K$ and $P \cap Vx \neq \emptyset\}$ and $z_x \in D \cap Vx$.
Suppose $A = A_1 \cap A_2 \subseteq D$. Then $P \subseteq AU$ and $P \subseteq UA$.

Proposition 3.3. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. If $G_{\mathfrak{R}}$ is a rough precompact, then every rough subgroup $H_{\mathfrak{R}}$ of $G_{\mathfrak{R}}$ is a precompact topological group.
Proof: Let U be an identity neighbourhood in $H_{\mathfrak{R}}$. Then there exists an identity neighbourhood V in $G_{\mathfrak{R}}$ such that $V \cap H_{\mathfrak{R}} \subseteq U \cap H_{\mathfrak{R}}$. Since $G_{\mathfrak{R}}$ is rough precompact, the rough subgroup $H_{\mathfrak{R}}$ is rough precompact. Then by lemma 3.2, there exists a finite subset $A \subseteq H_{\mathfrak{R}}$ such that
$H_{\mathfrak{R}} \subseteq A(V \cap H_{\mathfrak{R}})$ and $H_{\mathfrak{R}} \subseteq (V \cap H_{\mathfrak{R}})A$.
Since $V \cap H_{\mathfrak{R}} \subseteq V$ and $A \subseteq H_{\mathfrak{R}} \subseteq G_{\mathfrak{R}}$, $H_{\mathfrak{R}} \subseteq AV$ and $H_{\mathfrak{R}} \subseteq VA$. Consider an element $xh \in H_{\mathfrak{R}}$. Then $h \in AV$, so, $h = av$, for some $a \in A$ and $v \in V$. But we prove $h \in AU$, that is, we enough to prove that $v \in U$. Since $H_{\mathfrak{R}}$ is a rough subgroup of $G_{\mathfrak{R}}$ and
$h = av$, $v = a^{-1}h \in V \cap H_{\mathfrak{R}} \subseteq U \cap H_{\mathfrak{R}} \subseteq U$.
Therefore, $v \in U$, that is $h \in AU$ which implies $H_{\mathfrak{R}} \subseteq AU$. Similarly, $H_{\mathfrak{R}} \subseteq UA$. Hence, $H_{\mathfrak{R}}$ is a precompact topological group.

Proposition 3.4. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. Let S be a subset of $G_{\mathfrak{R}}$ contains a dense rough precompact subset. Then S is a rough precompact in $G_{\mathfrak{R}}$.
Proof: Let S be a subset of $G_{\mathfrak{R}}$ and D be a dense rough precompact subset of S. Let U be an identity open neighbourhood of $\overline{G_{\mathfrak{R}}}$. Then there exists a symmetric identity open neighbourhood V of $\overline{G_{\mathfrak{R}}}$ such that $VV \subseteq U$. Since D is a rough precompact subset of S, there exists a finite subset of F of $\overline{G_{\mathfrak{R}}}$ such that
$D \subseteq FV$ and $D \subseteq VF$.
Let $x \in S$. Then $D \cap xV \neq \emptyset$. So, consider an element $y \in D \cap xV$, that is $y \in D$ and $y \in xV$. Since $D \subseteq FV$, $y \in fV$, for some $f \in F$. But $y \in xV$ implies
$x \in yV^{-1} \subseteq fVV^{-1} \subseteq fU$.
Therefore, $S \subseteq FU$. Similarly, $S \subseteq UF$. Hence S is a rough precompact subset of $G_{\mathfrak{R}}$.

Proposition 3.5. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. Suppose A is a rough precompact subset of $\overline{G_{\mathfrak{R}}}$. Then there exists an identity neighborhood $V \subseteq \overline{G_{\mathfrak{R}}}$ such that $aVa^{-1} \subseteq W$, for every open neighborhood W of e in $\overline{G_{\mathfrak{R}}}$ and $a \in A$.
Proof: Let W be a neighborhood of e in $\overline{G_{\mathfrak{R}}}$. From lemma 2.12, there is an open set U of e in $G_{\mathfrak{R}}$ such that
$U \subseteq U^n \subseteq W$, for every $n \in \mathbb{N} - \{0\}$.
Since A is rough precompact, there exists a finite set $F \subseteq A$ such that
$A \subseteq UF$ and $A \subseteq FU$.
Consider $V = \bigcap_{x \in F} x^{-1}Wx$ is an identity open neighborhood in $G_{\mathfrak{R}}$. Now we choose an element $a \in A$ such that $a = ux$, for some $u \in U$ and $x \in F$. Hence,
$aVa^{-1} = uxVx^{-1}u^{-1} \subseteq uUu^{-1} \subseteq U^3 \subseteq W$, for any $a \in A$.

Proposition 3.6. If A is a closed rough precompact subset of a rough complete topological simple rough group $G_{\mathfrak{R}}$, then the subset A is compact.
Proof: Let $\mathcal{F}'$ be a rough ultrafilter on the set A. Consider a rough ultrafilter $\mathcal{F}$ on $G_{\mathfrak{R}}$ such that $\mathcal{F}' \subseteq \mathcal{F}$. Let U be an identity open neighborhood in $G_{\mathfrak{R}}$. Then there exist elements $a_1, a_2, \dots a_n \in G_{\mathfrak{R}}$ such that
$A \subseteq \bigcup_{i=1}^{n} a_i U$.

Since $A \in \mathcal{F}$ and $\mathcal{F}$ is a rough ultrafilter on $G_{\mathfrak{R}}$, $a_iU \in \mathcal{F}$, for some $i \leq n$. Therefore, by the definition of rough cauchy filter, $\mathcal{F}$ is a rough cauchy filter on $G_{\mathfrak{R}}$. Since $G_{\mathfrak{R}}$ is a rough complete topological simple rough group, $\mathcal{F}$ converges to some element x in $G_{\mathfrak{R}}$. Since $A \in \mathcal{F}' \subseteq \mathcal{F}$, every set in $\mathcal{F}$ intersects A. Therefore, $x \in cl(A)$, $cl(A)$ means closure of A. Since A is closed, $x \in A$. Hence $\mathcal{F}'$ converges to x in A which implies A is compact.

Theorem 3.7. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. A subset S of a topological group $\overline{G_{\mathfrak{R}}}$ is rough precompact in $G_{\mathfrak{R}}$ if and only if the closure of S in the rough complete group $\widehat{G_{\mathfrak{R}}}$ of $G_{\mathfrak{R}}$ is compact.
Proof: Suppose the closure of S, $cl(S)$ is compact in $\widehat{G_{\mathfrak{R}}}$ of $G_{\mathfrak{R}}$. Let U be an identity open neighbourhood in $\overline{G_{\mathfrak{R}}}$ and also let V be an identity open neighbourhood in $\overline{G_{\mathfrak{R}}}$ such that

$V \cap G_{\mathfrak{R}} \subseteq U \cap G_{\mathfrak{R}}$.
Since $cl(S)$ is compact and S is dense in $cl(S)$, there exists a finite subset F of S such that
$cl(S) \subseteq FV \cap VF$.
Let $x \in S \subseteq cl(S)$. Then $x = fv$, for some $f \in F$ and $v \in V$. Therefore,
$v = f^{-1}x \in F^{-1}S \subseteq S^{-1}S \subseteq G_{\mathfrak{R}}$.
Since $v \in V$,
$v \in G_{\mathfrak{R}} \cap V \subseteq U \cap G_{\mathfrak{R}} \subseteq U$.

Therefore, $x = fv \in FU$ which implies $S \subseteq FU$, similarly $S \subseteq UF$. Hence S is rough precompact. Conversely, suppose S is rough precompact in $G_{\mathfrak{R}}$. By proposition 3.4 and S is dense in $cl(S)$, $cl(S)$ is rough precompact. From proposition 3.6, $cl(S)$ is compact.

Corollary 3.8. Let P and Q be rough precompact subset of a topological simple rough group $G_{\mathfrak{R}}$. Then the sets P^{-1}, Q^{-1} and PQ are rough precompact in $G_{\mathfrak{R}}$.
Proof: Let $\widehat{G_{\mathfrak{R}}}$ be a rough complete of $G_{\mathfrak{R}}$. Then $cl(P)$ and $cl(Q)$ are compact by Proposition 3.7. Therefore, $\{cl(P)\}^{-1}, \{cl(Q)\}^{-1}$ and $cl(P)cl(Q)$ are compact. Since the sets P^{-1}, Q^{-1} and PQ are dense in $\{cl(P)\}^{-1}, \{cl(Q)\}^{-1}$ and $cl(P)cl(Q)$, by theorem 3.7, P^{-1}, Q^{-1} and PQ are rough precompact.

Proposition 3.9. Let P_i be a rough precompact subset of a topological simple rough group $G_{\mathfrak{R}}^{(i)}$, for each $i \in I$. Then the set $P = \prod_{i\in I} P_i$ is a rough precompact in the topological product $\prod_{i\in I} G_{\mathfrak{R}}^{(i)}$.
Proof: Let $\widehat{G_{\mathfrak{R}}}$ be a rough complete of $G_{\mathfrak{R}}$. By theorem 2.14, $\prod_{i\in I} \widehat{G_{\mathfrak{R}}^{(i)}}$ is rough precompact, then the closure of P is in $\widehat{G_{\mathfrak{R}}}$, $cl(P) = \prod_{i\in I} cl(P_i)$, where $cl(P_i)$ refers closure with respect to $\widehat{G_{\mathfrak{R}}^{(i)}}$, for each $i \in I$. Since P_i is rough precompact and by theorem 3.7, $cl(P_i)$ are compact. Therefore, $cl(P)$ is compact. Since P is dense in $cl(P)$, P is rough precompact.

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