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μ-Linear Transformations

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Abstract
A linear transformation is a particular kind of mapping between linear spaces that respect the basic operations of vector addition and scalar multiplication. This study formulates a theoretical framework for studying linear transformations in the theory of fuzzy linear spaces over fuzzy fields. For this, the notion of μ-linear transformations, which are linear transformations preserving the fuzzy structure of fuzzy linear spaces, is introduced. The notion of isomorphism of fuzzy linear spaces is also investigated.
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I. Introduction

An important stage in the evolution of the modern concept of uncertainty was the publication of a seminal paper by Lotfi A. Zadeh [9]. In his paper, Zadeh introduced a theory whose objects-fuzzy sets-are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree [3]. Through fuzzification, greater generality, higher expressive power and enhanced ability to model real world problems is gained. Growing number of fields and theories are undergrowing fuzzification. Within this expanding framework, a natural extension of classical algebraic systems under conditions of uncertainty has led to the development of fuzzy algebraic structures.

Wenxiang and Tu [1] introduced the notion of fuzzy linear spaces over fuzzy fields. This study is to investigate linear maps which preserve the fuzzy structure of fuzzy linear spaces, viz. μ-linear transformations. Section 2 of this paper gives some preliminaries and a brief summary of fuzzy fields and fuzzy linear spaces. The notion of μ-linear transformations is introduced in section 3. In section 4, isomorphism of fuzzy linear spaces is investigated. It is proved that the relation of isomorphism in the class of all fuzzy linear spaces is an equivalence relation.

II. Preliminaries

This section gives a brief summary of fuzzy fields and fuzzy linear spaces.

Definition 2.1 [1] Let X be a field and F a fuzzy set in X with membership function μ_F. Suppose the following conditions hold:
(i) $\mu_F\,(a+b) \geq \min\{\mu_F(a), \mu_F(b)\}$, $a, b \in X$
(ii) $\mu_F\,(a) = \mu_F(-a)$, $a \in X$
(iii) $\mu_F\,(ab) \geq \min\{\mu_F\,(a), \mu_F(b)\}$, $a, b \in X$
(iv) $\mu_F\,(a) = \mu_F\,(a^{-1})$, $a\,(\neq 0) \in X$.
Then F is called a *fuzzy field* in X and it is denoted by (F, X). Also (F, X) is called a *fuzzy field* of X.

Note that $\mu_F\,(0) \geq \mu_F\,(a)$ for all $a \in X$ and $\mu_F\,(1) \geq \mu_F\,(a)$ for all $a\,(\neq 0) \in X$.

Definition 2.2 [1] Let X be a field and (F, X) be a fuzzy field of X. Let Y be a linear space over X and V be a fuzzy set in Y with membership function μ_V. Suppose the following conditions hold:

(i) $\mu_V(x+y) \geq \min\{\mu_V(x), \mu_V(y)\}$, $x, y \in Y$
(ii) $\mu_V(x) = \mu_V(-x)$, $x \in Y$
(iii) $\mu_V(\lambda x) \geq \min\{\mu_F(\lambda), \mu_V(x)\}$, $\lambda \in X, x \in Y$
(iv) $\mu_F(1) \geq \mu_V(0)$.

Then (V, Y) is called a fuzzy linear space over (F, X).

Note that $\mu_V(0) \geq \mu_V(x)$ for all $x \in Y$.

Proposition 2.1 [1] *Let* (F, X) *be a fuzzy field of* X *and* Y *a linear space over* X*. Let* V *be a fuzzy set of* Y*. Then* (V, Y) *is a fuzzy linear space over* (F, X) *if and only if*

(i) $\mu_V(\lambda x + \mu y) \geq min\{\mu_F(\lambda) \wedge \mu_V(x), \mu_F(\mu) \wedge \mu_V(y)\}$, $\lambda, \mu \in X$ *and* $x, y \in Y$

(ii) $\mu_F(1) \geq \mu_V(x)$, $x \in Y$.

The Condition (i) in proposition 2 can be restated as

$$\mu_V(\lambda x + \mu y) \geq \min\{\mu_F(\lambda), \mu_F(\mu), \mu_V(x), \mu_V(y)\}.$$

Definition 2.3 [5] Let $Y \subseteq \mathbb{R}^n$, the n-dimensional Euclidean space. A fuzzy set V in Y is convex if $\mu_V(\lambda y_1 + (1 - \lambda)y_2) \geq \min\{\mu_V(y_1), \mu_V(y_2)\}$ for all y_1, $y_2 \in Y$ and for all $\lambda \in [0, 1]$.

In the next section, the idea of linear transformation between linear spaces is extended to membership preserving linear transformations between fuzzy linear spaces.

III. μ-Linear Transformations

This section provides a formal introduction to μ-linear transformations between fuzzy linear spaces and explores their essential structural characteristics.

If (V, Y) and (W, Z) are fuzzy linear spaces over a fuzzy field (F, X) and T is a linear transformation of Y into Z, then ($T(V)$, Z) and ($T^{-1}(W)$, Y) are fuzzy linear spaces over (F, X) [1]. Here $\mu_{T(V)}(Ty) = \sup\{\mu_V(y'): y' \in Y,\ Ty' = Ty\}$ and as a result, the following proposition.

Proposition 3.1 *Let Y and Z be linear spaces over the field X, T be a linear transformation of Y into Z and* (V, Y) *be a fuzzy linear space over* (F, X). If $y \in Y$, *then*

(i) $\mu_{T(V)}(Ty) \geq \mu_V(y')$ *for all* $y' \in Y$ *with* $Ty' = Ty$

(ii) $\mu_{T(V)}(Ty) = \mu_V(y)$ *if T is injective*

(iii) $\mu_{T(V)}(Ty) = \mu_V(0)$ *if* $y \in Ker\ T$.

Theorem 3.1 *Every linear transformation of a linear space Y into another linear space Z together with a fuzzy linear space in Z induces a fuzzy linear space in* Y.

Proof. Assume that Y and Z are linear spaces over field X.
Given a linear transformation T: Y →Z and a fuzzy a linear space (W, Z) over fuzzy field (F, X), consider $\mu_W \circ T: Y \to [0, 1]$, which satisfies:

(i) $(\mu_W \circ T)(ax + by) = \mu_W(T(ax + by)) = \mu_W(aTx + bTy)$

$$\geq \min\{\mu_F(a), \mu_F(b), \mu_W(Tx), \mu_W(Ty)\}$$

$$= \min\{\mu_F(a), \mu_F(b), (\mu_W \circ T)(x), (\mu_W \circ T)(y)\},\ a, b \in X \text{ and } x, y \in Y$$

(ii) $\mu_F(1) \geq \mu_W(0) \geq \mu_W(Ty) = (\mu_W \circ T)(y)$, $y \in Y$.

Consequently, the fuzzy set in Y with membership function $\mu_W \circ T$ is a fuzzy linear space in Y over the fuzzy field (F, X).

Definition 3.1 Let (V, Y) and (W, Z) be fuzzy linear spaces over the fuzzy field (F, X). If there exists a linear transformation $T : Y \to Z$ such that $\mu_W \circ T = \mu_V$, then (T, (V, Y), (W, Z), (F, X)) is said to be a μ-linear transformation.

Proposition 3.2 *If* (T, (V, Y), (W, Z), (F, X)) *is a* μ*-linear transformation*, *then*

(i) $\mu_W(0) = \mu_V(0)$

(ii) $\mu_V(y) = \mu_W(0)$ *for all* $y \in Ker\ T$.

Proof. For all $y \in Y$, $\mu_V(y) = (\mu_W \circ T)(y) = \mu_W(Ty)$. Hence

(i) $\mu_W(0) = \mu_W(T0) = \mu_V(0)$
(ii) If $y \in Ker\ T$, then $\mu_V(y) = \mu_W(Ty) = \mu_W(0)$.

Proposition 3.3 *If* $(T, (V, Y), (W, Z), (F, X))$ *is a* μ*-linear transformation, then*
(i) $T^{-1}(W) = V$
(ii) $T(V) \subseteq W$ *if* T *is injective.*

Proof. (i) For all $y \in Y$, $\mu_{T^{-1}(W)}(y) = \mu_W(Ty) = \mu_V(y)$.
(ii) Assume that T is injective. Let $z \in Z$.
If $T^{-1}(z) \neq \phi$, then there exists unique $y^* \in Y$ such that $Ty^* = z$. Therefore
$$\mu_{T(V)}(z) = \mu_{T(V)}(Ty*) = \mu_V(y*) = \mu_W(Ty*) = \mu_W(z).$$
If $T^{-1}(z) = \phi$, then $\mu_{T(V)}(z) = 0 \leq \mu_W(z)$.

Following corollary is immediate from proposition 3.3 and definitions of α-cut and strong α-cut.

Corollary 3.1 *If* $(T, (V, Y), (W, Z), (F, X))$ *is a* μ*-linear transformation and* $\alpha \in [0, 1]$, *then*
(i) $(T^{-1}(W))_\alpha = V_\alpha$ *and* $(T^{-1}(W))_{\alpha^+} = V_{\alpha^+}$
(ii) $(T(V))_\alpha \subseteq W_\alpha$ *and* $(T(V))_{\alpha^+} \subseteq W_{\alpha^+}$ *if* T *is injective.*

Proposition 3.4 *If* $(T, (V, Y), (W, Z), (F, X))$ *is a* μ*-linear transformation, then* $T(V_\alpha) \subseteq W_\alpha$ *and* $T(V_{\alpha^+}) \subseteq W_{\alpha^+}$.

Proof. $z \in T(V_\alpha) \Rightarrow z = Ty$ for some $y \in V_\alpha \Rightarrow \mu_W(z) = \mu_W(Ty) = \mu_V(y) \geq \alpha \Rightarrow z \in W_\alpha \Rightarrow T(V_\alpha) \subseteq W_\alpha$. Similarly, $T(V_{\alpha^+}) \subseteq W_{\alpha^+}$.

Proposition 3.5 *Let* $(T, (V, Y), (W, Z), (F, X))$ *be a* μ*-linear transformation and let* $y_1, y_2 \in Y$. *If* $Ty_1 = Ty_2$, *then* $\mu_V(y_1) = \mu_V(y_2)$.

Proof. $Ty_1 = Ty_2 \Rightarrow \mu_W(Ty_1) = \mu_W(Ty_2) \Rightarrow \mu_V(y_1) = \mu_V(y_2)$.

Proposition 3.6 *Let* Y *be a linear space in* $\mathbb{R}^m$ *and* Z *be a linear space in* $\mathbb{R}^n$. *If* $(T, (V, Y), (W, Z), (F, \mathbb{R}))$ *is a surjective* μ*-linear transformation and if* V *is convex in* Y, *then* W *is convex in* Z.

Proof. Let $0 \leq \lambda \leq 1$ and let $z_1, z_2 \in Z$. Since T is surjective, $z_1 = Ty_1$ *and* $z_2 = Ty_2$ for some $y_1, y_2 \in Y$ and so
$$\begin{aligned}\mu_W(\lambda z_1 + (1-\lambda)z_2) &= \mu_W(\lambda Ty_1 + (1-\lambda)Ty_2) = \mu_W\big(T(\lambda y_1 + (1-\lambda)y_2)\big)\\ &= \mu_V(\lambda y_1 + (1-\lambda)y_2) \geq \min\{\mu_V(y_1), \mu_V(y_2)\}, \text{since V is convex}\\ &= \min\{\mu_W(Ty_1), \mu_W(Ty_2)\} = \min\{\mu_W(z_1), \mu_W(z_2)\}.\end{aligned}$$
This means that W is convex in Z.

In the next section, the notion of isomorphism of fuzzy linear spaces is introduced.

IV. Isomorphism of Fuzzy Linear Spaces

Definition 4.1 If $(T, (V, Y), (W, Z), (F, X))$ is a μ-linear transformation and if T is bijective, then the fuzzy linear spaces (V, Y) and (W, Z) are said to be isomorphic and this isomorphism is denoted by $(V, Y) \cong (W, Z)$. Also, $(T, (V, Y), (W, Z), (F, X))$ is referred to as a fuzzy linear space isomorphism.

Proposition 4.1 *If* $(T, (V, Y), (W, Z), (F, X))$ *is a fuzzy linear space isomorphism, then* $T(V) = W$

Proof. Corresponding to each $z \in Z$, there exists a unique $y \in Y$ with $Ty = z$. As a result, by proposition 3.1(ii), $\mu_{T(V)}(z) = \mu_{T(V)}(Ty) = \mu_V(y) = \mu_W(Ty) = \mu_W(z)$, which implies $T(V) = W$.

Corollary 4.1 *If* $(T, (V, Y), (W, Z), (F, X))$ *is a fuzzy linear space isomorphism, then, for all* $\alpha \in [0, 1]$, $(T(V))_\alpha = W_\alpha$ *and* $(T(V))_{\alpha^+} = W_{\alpha^+}$.

Theorem 4.1 *The isomorphism relation on the class of all fuzzy linear spaces over a fuzzy field is an equivalence relation.*

Proof. *Reflexivity.*
For every fuzzy linear space (V, Y) over a fuzzy field (F, X), the identity mapping $I: Y \rightarrow Y$ serves as an isomorphism and $\mu_V \circ I = \mu_V$; consequently, (V, Y) is isomorphic to itself.
Symmetry.

If (V_1, Y_1) and (V_2, Y_2) are fuzzy linear spaces over a fuzzy field (F, X) and if $(V_1, Y_1) \cong (V_2, Y_2)$, then there exists an isomorphism $T : Y_1 \rightarrow Y_2$ such that $\mu_{V_1} = \mu_{V_2} o\, T$ so that $T^{-1} : Y_2 \rightarrow Y_1$ is an isomorphism and for every $z \in Y_2$, there exists a unique $y \in Y_1$ with $T^{-1}(z) = y$. Hence

$$\left(\mu_{V_1} o\, T^{-1}\right)(z) = \mu_{V_1}(T^{-1}(z)) = \mu_{V_1}(y) = \mu_{V_2}(Ty) = \mu_{V_2}(z).$$

This implies that $\mu_{V_2} = \mu_{V_1} o\, T^{-1}$. Therefore $(V_2, Y_2) \cong (V_1, Y_1)$.

Transitivity.

Let (V_1, Y_1), (V_2, Y_2) and (V_3, Y_3) be fuzzy linear spaces over (F, X). Assume that $(V_1, Y_1) \cong (V_2, Y_2)$ and $(V_2, Y_2) \cong (V_3, Y_3)$. Then there exist isomorphisms $T_1 : Y_1 \rightarrow Y_2$ such that $\mu_{V_2} o\, T_1 = \mu_{V_1}$ and $T_2 : Y_2 \rightarrow Y_3$ such that $\mu_{V_3} o\, T_2 = \mu_{V_2}$. In consequence, $T_2 \circ T_1 : Y_1 \rightarrow Y_3$ is an isomorphism with

$$\left(\mu_{V_3} o\, (T_2 \circ T_1)\right)(y) = \left(\left(\mu_{V_3} o\, T_2\right) o\, T_1\right)(y) = \left(\mu_{V_2} o\, T_1\right)(y) = \mu_{V_1}(y) \text{ for all } y \in Y_1$$

so that $\mu_{V_1} = \mu_{V_3} o (T_2 \circ T_1)$, which implies that $(V_1, Y_1) \cong (V_3, Y_3)$.

The following corollary is immediate.

Corollary 4.2 *The isomorphism relation on the class of all fuzzy linear spaces of a linear space over a fuzzy field is an equivalence relation.*

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