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On the accuracy of the deterministic approximation of the Markov model of investment and consumption optimization

Evgueni Gordienko[1], Adolfo Hernández-Iglesias[2]
[1]*(Department of Mathematics, Autonomous Metropolitan University, Mexico)*
[2]*(Department of Mathematics, Autonomous Metropolitan University, Mexico)*

Abstract: *This brief paper discusses the problem of long-term optimal consumption and investment. The corresponding model is formulated in terms of a discrete-time Markov control process.*
Rather than being concerned with the optimization of portfolio strategies, we are dealing with the accuracy of approximating the original control process with its deterministic version. In the case of small variances of returns on risky assets, we prove an inequality that gives an upper bound on the decrease in expected utility when implementing the approximation.
Key Word*: Optimal investment and consumption policy; Markov control process; Deterministic approximation of a process; Upper bound of the stability index.*

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I. Introduction

In modern applied and finance mathematics, there are many different *portfolio optimization models*. They are often defined in terms of optimal control of certain stochastic processes. (See, e.g., [2, 5, 8].)
This paper considers an investment and consumption optimization model that is set using a discrete-time Markov control process defined over an infinite planning horizon. The expected total discounted utility is used as a criterion for optimal asset allocation.
The purpose of this article is not to find an optimal investment policy, but to analyze the conditions under which the original stochastic control process can be approximated by a suitable deterministic control process. As a rule, for the latter, the search for optimal control policies turns out to be simpler. (See, e.g., the recent paper [6].)
Of course, the above approximation only makes sense if we assume that the random variables representing the returns of risky assets are sufficiently close to their mean values.
The main result of the paper is the inequality (3.7) (given in Section 3), which gives an upper bound on the loss in expected utility when the optimal investment policy found for the deterministic approximation is applied to the original control process.

II. The model and its deterministic approximation

We will consider a long-term time $t = 0,1,2, ...$ portfolio investment process:

$$X_t, \quad t = 0,1, \dots . \tag{2.1}$$

In (2.1), $X_0 \equiv x_0 > 0$ is a given *initial capital* of an investor, and X_t $(t = 1,2, ...)$ is his capital in the t-th period (say, month).
The model under consideration assumes the following scheme of consumption and investment.

(a) If in the beginning of the t-th period the capital is $X_{t-1} = x > 0$, then the investor spends (irrevocably) an amount $a_0^{(t)}x$ on *consumption*. Here, $a_0^{(t)} \in [0,1]$ is a controlled parameter.

(b) Suppose there are N assets available for investment $(N \geq 1)$. The remaining capital $y = (1 - a_0^{(t)})x$ is distributed among some of these assets in such a way that a share $a_k^{(t)}y$ is invested in the k-th asset. Here:

$$a_k^{(t)} \in [0,1],\ k = 1,2, \dots, N \quad \text{with} \quad \sum_{k=1}^{N} a_k^{(t)} = 1 \tag{2.2}$$

are also parameters that must be selected.

Therefore, the *control action* in this period is the choice of a vector $a^{(t)} = (a_0^{(t)}, a_1^{(t)}, \dots, a_N^{(t)})$ that satisfies (2.2) and $a_0^{(t)} \in [0,1]$.

The return of the k-th asset is given by a nonnegative integrable random variable $\xi_k^{(t)}$ with mean $m_k^{(t)} = E\xi_k^{(t)}$ ($k \in \{1,2,\dots,N\}$). If $\xi_k^{(t)} = m_k^{(t)}$ (a degenerate random variable), then the corresponding asset is *risk-free*, otherwise the asset is *risky*.

Thus, having chosen the control $a^{(t)}$, the agent's capital at the beginning of the t-th period is determined using the following equation:

$$X_t = \min\left\{(1 - a_0^{(t)})X_{t-1} \sum_{k=1}^{N} a_k^{(t)} \xi_k^{(t)}, \bar{x}\right\}, \quad t = 1,2,\dots, \tag{2.3}$$

where $\bar{x} > 0$ is a number representing a certain "natural limit of wealth".

In what follows we assume that $(\xi_1^{(1)}, \xi_2^{(1)}, \dots, \xi_N^{(t)}), (\xi_1^{(2)}, \xi_2^{(2)}, \dots, \xi_N^{(2)}), \dots$ are independent and identically distributed (i.i.d) random vectors, generically denoted as $\xi = (\xi_1, \dots, \xi_N)$.

Under the last assumption, the stochastic process in (2.3) forms a discrete-time *Markov control process* with state space $\mathfrak{X} = [0, \bar{x}]$ and compact admissible control (action) sets $A(x) \equiv A \overset{\text{def}}{=} [0,1] \times A_{st}$ ($x \in \mathfrak{X}$), where A_{st} is the set in $\mathbb{R}^N$ defined by (2.2) (see, e.g., [2, 7] for definitions). Any sequence $\pi = (a^{(1)}, a^{(2)}, \dots, a^{(t)}, \dots)$ with $a^{(t)} \in A$ is called a *control policy*, or simply *policy*. In our setting, each policy π sets a long-term investment strategy, and the main goal of the investor is to find (or to approximate) an *optimal policy* π_*, i.e., a policy such that for every $x_0 \in \mathfrak{X}$,

$$U(x_0, \pi_*) = \sup_{\pi \in \Pi} U(x_0, \pi), \tag{2.4}$$

where Π is the set of all policies, x_0 is an initial state of process (2.3), and $U(x_0, \pi)$ is a given *utility functional*. In our case, it is supposed that U is the expected total discounted utility defined as follows:

$$U(x_0, \pi) = E \sum_{t=1}^{\infty} \alpha^{t-1} u(X_{t-1}, a^{(t)}), \tag{2.5}$$

where $\pi = (a^{(1)}, a^{(2)}, \dots, a^{(t)}, \dots)$, $\alpha \in (0,1)$ is a given *discount factor*, and $u(x, a)$, $x \in \mathfrak{X}$, $a \in A$ is a specified *utility function* for one period (which usually depends on the current capital x and the consumption rate a_0).

Remark 2.1. If in (2.3) $N = 1$, and in (2.5) u is a power utility, then our optimization problem can be reduced to the problem of optimal management of the relationship between consumption and investment considered in [2].

As we noted in the introduction, we do not address the problem of finding an optimal investment policy, but rather attempt to estimate the accuracy of approximating the process (2.3) with its deterministic version:

$$\tilde{X}_t = \min\left\{(1 - a_0^{(t)})\tilde{X}_{t-1} \sum_{k=1}^{N} a_k^{(t)} m_k, \bar{x}\right\}, \tag{2.6}$$

where $m_k = E\xi_k$, $k = 1,2,\dots,N$.

III. Assumptions and the result

The set Π of control policies for process (2.6) is the same as for process (2.3). Similarly to (2.5), when applying a policy $\pi = (a^{(1)}, a^{(2)}, \dots)$ to the control process (2.6), the expected utility is

$$\widetilde{U}(x_0, \pi) = E \sum_{t=1}^{\infty} \alpha^{t-1} u(\tilde{X}_{t-1}, a^{(t)}), \tag{3.1}$$

and the optimal policy $\tilde{\pi}_*$ (of investment) for (2.6) is such that for all $x_0 \in \mathfrak{X}$, $\widetilde{U}(x_0, \tilde{\pi}_*) = \sup_{\pi \in \Pi} \widetilde{U}(x_0, \pi)$.

Using the below Assumption 3.1 and applying the well-known technique from [7], we can state that there exist stationary optimal policies π_* and $\tilde{\pi}_*$ for both control processes (2.3) and (2.6). (Stationary means that for a certain function $f: X \to A$, $a^{(t)} = f(X_{t-1})$ for all $t = 1,2, \dots$.)

Now suppose that the investor cannot find the optimal policy π_* for the original process (2.3), whereas he can do so for a simpler deterministic process (2.6), i.e., to find $\tilde{\pi}_*$ (see, e.g., [6] on optimization of deterministic discrete-time control processes). When the random returns ξ_k in (2.3) do not deviate much from their mean values m_k, one can try $\tilde{\pi}_*$ as a reasonable approximation to π_*, that is, use $\tilde{\pi}_*$ to control process (2.3). With such a replacement we encounter a decrease in expected utility, expressed by the following *stability index* (see, e.g., [3, 4]):

$$\Delta(x_0) = U(x_0, \pi_*) - U(x_0, \tilde{\pi}_*), \quad x_0 \in \mathfrak{X}. \tag{3.2}$$

The below Theorem 3.1 provides an upper bound for $\Delta(x_0)$.

Let $\beta \in (\alpha, 1)$ and $\gamma \in (0,1)$ be arbitrary but fixed in what follows numbers, and $u(x, a)$ a given one-period utility function.

Assumption 3.1.

(a) There are constants c and L_0 such that:

1. $|u(x,a)| \le c(x^\gamma + 1), x \in \mathfrak{X}, a \in A$; (3.3)
2. $|u(x,a) - u(y,a)| \le L_0|x-y|, x, y \in \mathfrak{X}, a \in A$. (3.4)

(b) For all $a_1, \ldots, a_N$ such that $a_k \in [0,1]$, $k = 1,2, \ldots, N$ and $\sum_{k=1}^{N} a_k = 1$, the inequality

$$E\left[\sum_{k=1}^{N} a_k\, \xi_k\right]^\gamma \le \frac{\beta}{\alpha} \tag{3.5}$$

holds.

(c) In (2.6) $m_k = E\xi_k \le 1$ for all $k = 1,2, \ldots, N$. (3.6)

(d) For every $k = 1,2, \ldots, N$, the random variable ξ_k is bounded.

Theorem 3.1. *Under Assumption 3.1, for every initial state $x_0 \in [0, \bar{x}]$, the following inequality holds (see (3.2)):*

$$\Delta(x_0) \le C(x_0) E\left[\max_{1 \le k \le N} |\xi_k - m_k|\right], \tag{3.7}$$

where

$$C(x_0) = \frac{2\alpha\bar{x}\max\{1, L_0\}}{1-\alpha}\left[\frac{1}{1-\alpha} + \frac{\alpha c}{(1-\beta)^2}(x^\gamma + 1)\right]. \tag{3.8}$$

Remark 3.1.

(a) In some circumstances, the constant $C(x_0)$ in (3.7), (3.8) may be large. In any case, inequality (3.7) allows us to assert that if $E\left[\max_{1 \le k \le N} |\xi_k - m_k|\right]$ approaches zero, then the additional losses in utility disappear.

(b) Since the control set A is compact, the condition (3.3) is satisfied if, for example, $u(x,a) = u_1(a)$ (for instance, $u_1(a) = b a_0^\kappa$) and the last function is continuous. Also, in this case, (3.4) holds for any $u_1(a)$.

(c) Let us examine a simple particular case when conditions (3.5) and (3.6) are fulfilled. Let in (2.3), (2.5) $N = 1$ and $\xi \sim \text{Unif}(1-\varepsilon, 1+\varepsilon)$ (with some small $\varepsilon > 0$), and in Assumption 3.1, $\gamma = 0.5$. Then $m_1 = E\xi_1 = 1$, and it can easily be shown that $E(\xi_1)^{1/2} < 1$. The latter leads to inequality (3.5). Also, the second factor on the right-hand side of inequality (3.7) is $\varepsilon/2$.

The proof of Theorem 3.1

To prove inequality (3.7) we will use the main part of the Theorem from [1]. To be able to apply the stability inequality (2.3) from [1], we need to verify the hypotheses of the above-mentioned theorem.

First, we introduce notation (see (2.3) and (3.3)):

$$W(x) \stackrel{\text{def}}{=} c(x^\gamma + 1), x \in \mathfrak{X}, \tag{3.9}$$

$$F_1(x,a,s) \stackrel{\text{def}}{=} (1-a_0)x\sum_{k=1}^{N} a_k s_k\,, \tag{3.10}$$

where $(a_0, a_1, \ldots, a_N) \in A, x \in \mathfrak{X}$ and $s_k \ge 0, k = 1,2, \ldots, N$. Also:

$$F(x,a,s) \stackrel{\text{def}}{=} \min\{F_1(x,a,s), \bar{x}\} \tag{3.11}$$

Assumption 1, (a) in [1] is met due to our Assumption 3.12, (a).

To verify Assumption 1, (b) from [1], we have to check the following inequality:

$$EW[F(x,a,\xi)] \le \frac{\beta}{\alpha} W(x) \tag{3.12}$$

(for all $x \in \mathfrak{X}, a \in A$).

Since the function W in (3.9) is increasing and $F \le F_1$, we can replace in (3.12) F with F_1. Then in view of (3.9), (3.10),

$$EW[F_1(x,a,\xi)] = Ec\left\{(1-a_0)^\gamma x^\gamma \left(\sum_{k=1}^{N} a_k \xi_k\right)^\gamma + 1\right\}$$

$$\le c\left\{x^\gamma E\left(\sum_{k=1}^{N} a_k \xi_k\right)^\gamma + 1\right\}$$

$$\leq c\left\{x^{\gamma}\frac{\beta}{\alpha}+\frac{\beta}{\alpha}\right\}$$
$$\leq \frac{\beta}{\alpha}c(x^{\gamma}+1)=\frac{\beta}{\alpha}W(x).$$

Here, we used inequality (3.5) in Assumption 3.1.
Let us now verify Assumption 1, (c) in [1]. To do this, we need to prove that for every bounded continuous function $\varphi: \mathfrak{X} \times A \to \mathbb{R}$, the function $\bar{\varphi} \overset{\text{def}}{=} E\varphi[F(x,a,\xi)]$ is continuous on $\mathfrak{X} \times A$. In view of (3.11), it is enough to check the continuity of the function $\bar{\bar{\varphi}} \overset{\text{def}}{=} E\varphi[F_1(x,a,\xi)]$.
Since for each $\xi = s$, the function $F_1(x,a,s)$ is continuous (in x and a) and u is bounded, the desired continuity follows from the dominated convergence theorem. Also, the same theorem and Assumption 3.1, (d) implies the continuity (in a and x) of the function $EW[F_1(x,a,\xi)] = Ec\{(1-a_0)^{\gamma}x^{\gamma}(\sum_{k=1}^{N} a_k\xi_k)^{\gamma}+1\}$.
Let's move onto testing Assumption 2 in [1]. The item (a) of the latter is the same as Assumption 3.1, (a); (1).
Let $m = (m_1, m_2, \ldots, m_N)$. Then,

$$|F_1(x,a,m)-F_1(y,a,m)| \leq (1-a_0)\sum_{k=1}^{N} a_k m_k |x-y| \leq \max_{(a_1,\ldots,a_N)\in A_1}\sum_{k=1}^{N} a_k m_k \leq 1$$

due to Assumption 3.1, (c).
Finally, we need to establish the Lipschitz property with respect to s of the function $F(x,a,s)$ (see Assumption 2, (c) in [1]). Again, due to (3.11) we can work on the function F_1 in (3.10). Fixing any $x \in [0,\bar{x}]$ and $a \in A$, we get

$$|F_1(x,a,s)-F_1(x,a,s')| = (1-a_0)x\left|\sum_{k=1}^{N} a_k s_k - \sum_{k=1}^{N} a_k s'_k\right|$$
$$\leq \bar{x}\sum_{k=1}^{N} a_k|s_k - s'_k|$$
$$\leq \bar{x}\max_{1\leq k\leq N}|s_k - s'_k| = \bar{x}r(s,s'),$$

where

$$r(s,s') \overset{\text{def}}{=} \max_{1\leq k\leq N}|s_k - s'_k|. \tag{3.13}$$

Now we are ready to apply inequality (2.3) in [1], which, using our specific notation, leads to inequality (3.7). The right-hand side of (3.7) exploits the metric r in (3.13).

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