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Necessary And Sufficient Condition For Hypercyclicity Of Basic Elementary Operator

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Abstract:
We present a necessary and sufficient condition for hypercyclicity of a basic elementary operator
Keywords: *Hypercyclicity, Hypercyclicity criterion*

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I. Introduction

The property of hypercyclicity has been established to be equivalent to transitivity. Hypercyclicity of backward shift operator has been established. Recent research has shown that a bounded linear operator is hypercyclic using Chan's approach. It has also been shown that a basic elementary operator is hypercyclic and that it satisfies the three properties of hypercyclity criterion. This paper gives a necessary and sufficient condition for hypercyclicity of a basic elementary operator.

An operator $S \in L(X)$ is termed as *hypercyclic* if there is a vector $x \in X$ such that its S-orbit densely covers the space X. The corresponding vector x is known as S- hypercyclic.

If X is a topological vector space and S an operator in $L(X)$. Then S meets the *hypercyclicity criterion* if there is an increasing sequence of integers n_k, two dense sets $D_1, D_2 \in X$, and a sequence of maps $T_{nk}: D_2 \to X$ such that;
i) $S^{nk}(x)$ and T_{nk} converges to 0 for any $x \in D_1$ and $y \in D_2$ respectively.
ii)$S^{nk}T_{nk}(y)$ converges to y for any $y \in D_2$

Theorem 1.1[Kit,1999]

If a countable SOT- dense subset C of $A(K)$ exists and a linear mapping $B: A(K) \to A(K)$, with LB being the identity mapping on $A(K)$ and $\lim_{n\to\infty} \|B^n(U)\| = 0$ for each operator U in C, then a bounded linear mapping $L: A(K) \to A(K)$ is hypercyclic. Additionally, $\lim_{n\to\infty} \|L^n(U)\| = 0$.

Sophie [2005] determined Orbits of backshift operators by proving theorem 1.2. He also gave the conditions for hypercyclicity of a backward shift operator .

Theorem 1.2[Sophie,2005**]**

If (w_k) is a decreasing sequence of positive weights such that $k(w_1, w_2, \dots, w_n)^{\frac{1}{k}}$ tends to zero as n tends to infinity and A is the backward shift operator on $l_p, 1 \leq p < +\infty, c_o$ with weight$(w_k), 1 + A$ is a mixing operator which does not obey the Kitai's Criterion. The operator $1 + A$ exhibits a strange behavior when the weights are very small which means that no vector with an infinite support can have an orbit that grows polynomially.
Yousefi and Rezaei[2005] proved the following;

Theorem 1.3[Yousefi and Rezaei,2005**]**
For an operator $S \in A(K)$ the following are equivalent;
i.S satisfies the hypothesis of the hypercyclicity criterion
ii.For each pair U, V of non-void open subsets of K, and each neighborhood W of zero, $S^n U \cap W \neq \emptyset$ and $S^n W \cap V \neq \emptyset$

On their part Farrukh and Octabek [2017] showed that any unilateral, backward shift $B_\propto$ is hypercyclic iff $\lim sup_{n\to\infty} \prod_{i=1}^{n} |a_i| = \infty$

II. Necessary And Sufficient Condition For Hypercyclicity Of Basic Elementary Operator

Theorem 2.1 and 2.2 gives the results;

Theorem 2.1

Let $M_{T,S}: A(K) \to A(K)$be a basic elementary operator . $M_{T,S}$is said to be hypercyclic if there exist a linear mapping $W: A(K) \to A(K)$ and a countable SOT-dense subset V of $A(K)$ such that $M_{T,S}W$ is the identity mapping on $A(K)$ and for each operator D in V ,

$$\lim_{n\to\infty} \|W^n(D)\| = 0$$

and

$$\lim_{n\to\infty} \left\|M_{T,S}(D)\right\| = 0$$

Proof

Let $C = \{B_j: j \geq 1\}$ represent the countable SOT-dense subset C. A series of integers $n_j: j \geq 1$can be found. Determine a positive number n_2 such that the following three inequalities hold, given n_1 being a positive integer such that $\|W^{n1}(B_1)\| < 2^{-1}$

$\|W^{n1+n2}(B_2)\| < 2^{-1}2^{-2}$………………..(1)

$\|W^{n2}(B_2)\| < 2^{-2}$ … … … … … ….………………(2)

$\left\|{M_{T,S}}^{n2}(B_2)\right\| < 2^{-2}$………………………(3)

n_1, n_2 allows us to choose a positive integer n_3 such that

$$\|W^{n_1+n_2+n_3}(B_3)\| < 2^{-1}2^{-2}2^{-3}$$
$$\|W^{n_2+n_3}(B_3)\| < 2^{-2}2^{-3}$$
$$\|W^{n_3}(B_3)\| < 2^{-3}$$

And also

$$\left\|{M_{T,S}}^{n_2+n_3}(B_1)\right\| < 2^{-2}2^{-3}$$
$$\left\|{M_{T,S}}^{n_3}(B_2)\right\| < 2^{-3}$$

We can find a positive integer n_j such that, for any integers $m = 1,2, \dots, j$ we can generalise this so that we have;

$\|W^{n_m+n_{m+1}+\cdots+n_j}(B_k)\| < 2^{-m}2^{-m-1} \dots 2^{-j}$ … … … … … . (4)

moreover, for every integer $l = 2,3, \dots j$

$\left\|{M_{T,S}}^{n_l+n_{l+1}+\cdots+n_j}(B_{l-1})\right\| < 2^{-l}2^{-l-1} \dots 2^{-j}$……..(5)

Our definition of B with respect to the sequence n_j is;

$$B = \sum_{J=1}^{\infty} W^{n_1+n_2+\cdots+n_j}\, B_j$$

wherein each summand fulfils

$$\left\|W^{n_1+n_2+\cdots+n_j}(B_j)\right\| < 2^{-1}2^{-2} \dots 2^{-j}$$

As a result, the series equation defining B in $A(K)$ is totally convergent.

Next, by first noting the following, we demonstrate that B is a hypercyclic vector of $M_{T,S}$for each integer $m \geq 2$

$${M_{T,S}}^{n_1+n_2+\cdots+n_m}(B) = \sum_{J=1}^{\infty} {M_{T,S}}^{n_1+n_2+\cdots+n_m} W^{n_1+n_2+\cdots+n_j}\, B_j$$

due to $M_{T,S}$ boundedness

$$= \sum_{j=1}^{m-1} {M_{T,S}}^{n_{j+1}+\cdots+n_m} + (B_j) + B_m + \sum_{j=m+1}^{\infty} W^{n_1+n_2+\cdots+n_j}(B_j)$$

And therefore,

$$\left\|{M_{T,S}}^{n_1+n_2+\cdots+n_m}(B) - B_m\right\| \leq \sum_{j=1}^{m-1} \left\|{M_{T,S}}^{n_{j+1}+\cdots+n_m} + (B_j)\right\| + \sum_{j=m+1}^{\infty} \left\|W^{n_1+n_2+\cdots+n_j}(B_j)\right\|$$

$$\leq \sum_{j=1}^{m-1} 2^{-j-1} \dots 2^{-m} + \sum_{j=m+1}^{\infty} 2^{-m-1} \dots 2^{-j} < 2^{-m} \sum_{j=0}^{\infty} 2^{-j} + 2^{-m-1} \sum_{j=0}^{\infty} 2^{-j} = 2^{-m+1} + 2^{-m}$$

and hence

$\lim_{m\to\infty} \left\|{M_{T,S}}^{n_1+n_2+\cdots+n_m}(B) - B_m\right\| = 0$………….(6)

Here, we employ equation (6) above to demonstrate that an operator of the kind exists in every basic SOT-open set D.

$${M_{T,S}}^{n_1+n_2+\cdots+n_\propto}(B)$$

for any integer$\propto$, and so $orb(T,S)$ is SOT-dense in $A(K)$ The strong operator topology formulation allows D to be written as ;

$$D = D\left(U_{0,}\epsilon; h_1, h_2, \ldots, h_N\right)$$
$$= \{U \in A(K): \|(U - U_0)h_i\| < \epsilon, \forall i = 1, \ldots, N\}$$

where $\epsilon > 0, U_0$ is an operator in $A(K)$ and $h_1, h_2, \ldots h_N$are N vectors in K.

If all h_i are zero vectors, then $D = A(K)$ so we can conclude that at least one h_i is non-zero in our proof. Let E be the maximum value of $\|h_i\|$for $1 \le i \le N$. Equation (6) provides an integer M such that for all integers $m \ge M$ the inequality

$\left\|{M_{T,S}}^{n_1+n_{2+\cdots+n_m}}(B) - B_m\right\| < (2E)^{-1}\epsilon$ holds true.

If $1 \le i \le N$ and $m \ge M$ then

$\left\|{M_{T,S}}^{n_1+n_{2+\cdots+n_m}}(B)h_i - B_m(h_i)\right\| < (2E)^{-1}\epsilon\|h_i\| \le \frac{\varepsilon}{2}$ ………….(7)

Again, the set C densely spans $A(K)$ in the strong operator topology (SOT), so the set $\{B_m: m \ge M\}$ also densely spans $A(K)$ in the SOT. Therefore, we can select an integer $\propto \ge M$in such a way that

$B_\alpha \in D\left(U_{0,}\frac{\varepsilon}{2}; h_1, h_2, \ldots, h_N\right)$ for every $i = 1,2, \ldots, N$

$\|(B_\alpha - U_0)h_i\| < \frac{\varepsilon}{2}$…………..(8)

Consequently, for that specific integer $\alpha$and for every h_i

$$\left\|{M_{T,S}}^{n_1+n_{2+\cdots+n_\alpha}}B(h)_i - U_o(h_i)\right\| \le \left\|{M_{T,S}}^{n_1+n_{2+\cdots+n_\alpha}}(B)h_i - B_\alpha(h_i)\right\|\|B_\alpha(h_i) - U_o(h_i)\|$$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon <$

Using Inequalities (7) and (8), we can conclude that.

$\left({M_{T,S}}^{n_1+n_{2+\cdots+n_\alpha}}\right)(S) \epsilon V$and the proof is now complete.

The following theorem outlines a requirement that must be met for the basic elementary operator to exhibit hypercyclicity.

Theorem 2.2

If $M_{T,S}: A(K) \to A(K)$is hypercyclic , then for every scalar β the mapping $M_{T,S} - \beta$ has an an SOT-dense range.

Proof

If ${M_{T,S}}^*$ is a Banach space adjoint of the basic elementary operator $M_{T,S}$ then every non-zero SOT linear functional Z is not hypercyclic if $\exists$ a scalar $\beta: {M_{T,S}}^* = \beta Z$

To show this;

$$Z\left(M_{T,S}^n\right)W = \left(M_{T,S}^{*n}Z\right)(W) = \beta^n Z(W)$$

for any operator $W \epsilon A(K)$and any positive integer n.

This means that $orb(W, M_{T,S})$ is not SOT dense in $A(K)$. Therefore if $M_{T,S}$ is hypercyclic, then for any non-zero SOT- continous linear functional Z on $A(K)$ and for every scalar β then there exists $W \in A(K)$ such that $Z(M_{T,S} - \beta)W \neq 0$

So, because $A(K)$ is a locally convex topological vector space in SOT, it follows that $\left(M_{T,S} - \beta\right)A(K)$ is SOT dense in $A(K)$. If the operator $M_{T,S}$ possesses a single hypercyclic vector W, then it is necessary for $M_{T,S}$ to have a densely scattered set of hypercyclic vectors under the strong operator topology (SOT) and $\left(M_{T,S}^n\right)W$ is hypercyclic vector for $M_{T,S}$ to see this, we notice that the set;

$$\left\{W, M_{T,S}, M_{T,S}^2, M_{T,S}^3, \ldots\right\}$$

is SOT dense in $A(K)$ which implies that the set $\{\left(M_{T,S}^n\right)W, \left(M_{T,S}^{n+1}\right)W, \left(M_{T,S}^{n+2}\right)W, \ldots\}$ is also SOT dense.

III. Conclusion

Necessary and sufficient conditions for hypercyclicity of basic elementary operator has been established using Chan's approach. Necessary and sufficient conditions for hypercyclicity of other operators can be attempted using similar approach.

References

[1]. Ansari, S. (1997). Existence Of Hypercyclic Operators On Topological Vector Spaces. J. Funct. Anal., 148:384-390.

[2]. Chan, K. (2001). The Density Of Hypercyclic Operators On A Hilbert Space. J. Operator Theory,47:131–143.

[3]. Clifford, O. (2019). Dynamics Of Generalised Derivations And Elementary Operators. Complex Analysis And Operator Theory, 13:257–274.

[4]. Farrukh, M. And Octabek, K. (2017). Hyper-Cyclic And Supercyclic Linear Operators On Non-Archmedian Vector. Bulletin Belgium Math. Soc., 12:10–15.

[5]. Grivaux, S. (2005). Hypercyclic Operators,Mixing Operators And Bounded Step Theorem. J.Math. Anal.Appl., 54:147–168.

[6]. Kitai, C. (1982). Invariant Closed Sets For Linear Operators. Ph.D Thesis,Univ. Of Toronto. Montes-Rodriguez, A. (1996). Banach Spaces Of Hyper-Cyclic Vectors. Michigan Math. J., 43:419–436.

[7]. Rezaei, H. (2005). Notes On The Hypercyclic Operator. Journal Of Mathematical Extension., 1:39–44.

[8]. Sophie, G. (2005). Hyper-Cyclicity Operators,Mixing Operators And Bounded Step Theorem.Semantic Scolars, 6:7–10.
[9]. Yousefi, B. And Rezaei, H. (2005). Some Necessary And Sufficient Conditions For Hypercyclicity Criterion. Proc.Indian.Acad.Sci., 115):209–216.
[10]. Yousefi, B. And Rezaei, H. (2008). Hyper-Cyclic Property Of Weighted Operators. Proc.Amer.Math.Soc, 135:3263–3271.