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Dominating Sets and Domination Polynomials of Square Of Cycles

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Abstract: *Let G = (V, E) be a simple graph. A set S $\subseteq$ V is a dominating set of G, if every vertex in V-S is adjacent to atleast one vertex in S. Let* C_n^2 *be the square of the Cycle* C_n *and let* $D(C_n^2, i)$ *denote the family of all dominating sets of* C_n^2 *with cardinality i. Let* $d(C_n^2, i)$ = $|D(C_n^2, i)|$. *In this paper, we obtain a recursive formula for* $d(C_n^2, i)$ *. Using this recursive formula, we construct the polynomial,* $D(C_n^2, x) = \sum_{i=\lceil \frac{n}{5} \rceil}^{n} d(C_n^2, i)x^i$ *, which we call domination polynomial of* C_n^2 *and obtain some properties of this polynomial.*
Keywords: *domination set, domination number, domination polynomials*.

I. Introduction

Let G = (V,E) be a simple graph of order |V| = n. For any vertex v$\in$V, the open neighbourhood of v is the set N (v) = {u$\in$V/uv$\in$E} and the closed neighbourhood of v is the set N[v] = N (v) $\cup${v}. For a set S $\subseteq$ V, the open neighbourhood of S is N(S) = $\cup_{v\in S}$ N (v) and the closed neighbourhood of S is N[S] = N(S) $\cup$ S. A set S $\subseteq$ V is a dominating set of G, if N[S] = V, or equivalently, every vertex in V-S is adjacent to atleast one vertex in S. The domination number of a graph G is defined as the minimum size of a dominating set of vertices in G and it is denoted as $\gamma(G)$ .A cycle can be defined as a closed path, and is denoted by C_n .

1.1. Definition

The square of a graph: The 2nd power of a graph with the same set of vertices as G and an edge between two vertices if and only if there is a path of length atmost 2 between them.
Let C_n^2 be the square of the cycle C_n (2nd power) with n vertices. Let $D(C_n^2, i)$ be the family of dominating sets of the graph C_n^2 with cardinality i and let $d(C_n^2, i)$ = | $D(C_n^2, i)$ |. We call the polynomial

$D(C_n^2, x) = \sum_{i=\lceil \frac{n}{5} \rceil}^{n} d(C_n^2, i)x^i$ the domination polynomial of the graph C_n^2.

In the next section, we construct the families of the dominating sets of the square of cycles by recursive method.

As usual we use $\lfloor x \rfloor$ for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x. Also, we denote the set {1, 2… n} by [n], throughout this paper.

II. Dominating Sets Of Square Of Cycles

Let $D(C_n^2, i)$ be the family of dominating sets of C_n^2 with cardinality i. We investigate the dominating sets of C_n^2. We need the following lemma to prove our main results in this section.

Lemma.2.1

$$\gamma(C_n^2) = \left\lceil \frac{n}{5} \right\rceil$$

By Lemma2.1 and the definition of domination number, one has the following lemma:

Lemma.2.2

$$D(C_n^2, i) = \Phi \text{ if and only if } i \succ n \text{ or } i \prec \left\lceil \frac{n}{5} \right\rceil$$

A simple path is a path in which all its internal vertices have degree two, and the end vertices have degree one. The following lemma follows from observation.

Lemma 2.3 [2]

If a graph G contains a simple path of length 5k -1, then every dominating set of G must contain at least k vertices of the path.

In order to find a dominating set of C_n^2, with cardinality i, we only need to consider $D(C_{n-1}^2, i-1), D(C_{n-2}^2, i-1), D(C_{n-3}^2, i-1), D(C_{n-4}^2, i-1)\, and\, D(C_{n-5}^2, i-1)$ .The families of these dominating sets can be empty or otherwise. Thus, we have eight combinations, whether these five families are empty or not. Two of these combinations are not possible (Lemma 2.5 (i), & (ii)).
Also, the combination that

$$D(C_{n-1}^2, i-1) = D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = D(C_{n-5}^2, i-1) = \Phi$$

does not need to be considered because it implies that $D(P_n^2, i) = \Phi$ (See lemma 2.5 (iii)). Thus we only need to consider five combinations or cases. We consider those cases in theorem (2.7).

Lemma 2.4

If Y $\in D(C_{n-6}^2, i-1)$, and there exists x $\in$ [n] such that Y $\cup$ {x} $\in D(C_n^2, i)$,
then Y $\in D(C_{n-5}^2, i-1)$.

Proof:

Suppose that Y $\notin D(C_{n-5}^2, i-1)$.

Since Y $\in D(C_{n-6}^2, i-1)$, Y contains at least one vertex labeled n–6, n–7 or n–8.

If n – 6 $\in$ Y, then Y $\in D(C_{n-5}^2, i-1)$, a contradiction. Hence n –7 or n – 8 $\in$ Y, but then in this case, Y $\cup$ {x} $\notin D(C_n^2, i)$, for any x$\in$ [n], also a contradiction.

Lemma 2.5

i) If $D(C_{n-1}^2, i-1) = D(C_{n-3}^2, i-1) = \Phi$ then $D(C_{n-2}^2, i-1) = \Phi$

ii) If $D(C_{n-1}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) \neq \Phi$ then $D(C_{n-2}^2, i-1) \neq \Phi$,

iii) If $D(C_{n-1}^2, i-1) = \Phi$, $D(C_{n-2}^2, i-1) = \Phi$, $D(C_{n-3}^2, i-1) = \Phi$, $D(C_{n-4}^2, i-1) = \Phi$, $D(C_{n-5}^2, i-1) = \Phi$, then $D(C_n^2, i) = \Phi$

Proof

i) Since $D(C_{n-1}^2, i-1) = D(C_{n-3}^2, i-1) = \Phi$, by lemma 2.2, i -1 $\succ$ n-1 (or) i -1 $\prec \left\lceil \frac{n-3}{5} \right\rceil$.

ii) In either case, we have $D(C_{n-2}^2, i-1) = \Phi$.

iii) Suppose that $D(C_{n-2}^2, i-1) = \Phi$, so by Lemma 2.2, we have i – 1 $\succ$ n – 2 or i – 1 $\prec \left\lceil \frac{n-2}{5} \right\rceil$.

If i -1 $\succ$ n – 2, then i – 1 $\succ$ n – 3. Therefore $D(C_{n-3}^2, i-1) = \Phi$, a contradiction.

iv) Suppose that $D(C_n^2, i) \neq \Phi$, Let Y $\in D(C_n^2, i)$). Thus, atleast one vertex labeled n or n – 1 or n – 2 is in Y. If n $\in$ Y, then by Lemma 2.3, at least one vertex labeled n – 1, n – 2, n – 3, n – 4 or n – 5 is in Y.
If n – 1 $\in$ Y or n – 2 $\in$ Y, then Y – {n} $\in D(C_{n-1}^2, i-1)$, a contradiction. If n – 3 $\in$ Y,

then Y – {n} $\in D(C_{n-2}^2, i-1)$, a contradiction. Now, suppose that n – 1 $\in$ Y. Then, by Lemma 2.3, at least one vertex labeled n – 2, n – 3, n – 4, n – 5 or n – 6 is in Y. If n – 2 $\in$ Y or n–3 $\in$ Y, then Y – {n – 1} $\in D(C_{n-2}^2, i-1)$ a contradiction. If n – 4 $\in$ Y, then Y – {n – 1} $\in D(C_{n-3}^2, i-1)$, a contradiction. If n – 5 $\in$ Y, then Y – {n – 1} $\in D(C_{n-4}^2, i-1)$, a contradiction. If n – 6 $\in$ Y, then Y – {n – 1} $\in D(C_{n-5}^2, i-1)$, a contradiction. Therefore, $D(C_n^2, i) = \Phi$.

Lemma 2. 6

If $D(C_n^2, i) \neq \Phi$,then

i) $D(C_{n-1}^2, i-1) = D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = \Phi$ and $D(C_{n-5}^2, i-1) \neq \Phi$ if and only if n=5k and i=k for some k $\in$ N;

ii) $D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = D(C_{n-5}^2, i-1) = \Phi$ then $D(C_{n-1}^2, i-1) \neq \Phi$ if and only if i = n;

iii) $D(C_{n-1}^2, i-1) = \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) \neq \Phi$, $D(C_{n-4}^2, i-1) \neq \Phi$, $D(C_{n-5}^2, i-1) \neq \Phi$, if and only if n = 5k+2 and i = $\left\lceil \frac{5k+2}{5} \right\rceil$ for some k$\in$N;

iv) $D(C_{n-1}^2, i-1) \neq \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) \neq \Phi$, $D(C_{n-4}^2, i-1) \neq \Phi$, and $D(C_{n-5}^2, i-1) = \Phi$ if and only if i = n – 3.

v) $D(C_{n-1}^2, i-1) \neq \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) \neq \Phi$, $D(C_{n-4}^2, i-1) \neq \Phi$, and $D(C_{n-5}^2, i-1) \neq \Phi$ if and only if $\left\lceil \frac{n-1}{5} \right\rceil$ +1 $\leq$ i $\leq$ n – 4.

Proof

i) ($\Rightarrow$)

Since $D(C_{n-1}^2, i-1)) = D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = \Phi$, by lemma 2.2,

i – 1 $\succ$ n – 1 or i – 1 $\prec \left\lceil \frac{n-2}{5} \right\rceil$

If i – 1 $\succ$ n – 1, then i $\succ$ n and by lemma 2.2, $D(C_n^2, i) = \Phi$, a contradiction.

So i $\prec \left\lceil \frac{n-2}{5} \right\rceil$ + 1, and since $D(C_n^2, i) \neq \Phi$, we have $\left\lceil \frac{n}{5} \right\rceil \leq$ i $\leq \left\lceil \frac{n-2}{5} \right\rceil$+ 1, which implies that n = 5k and i = k for some k$\in$N.

($\Leftarrow$) If n = 5k and i = k for some k$\in$N, then by lemma 2.2,

$D(C_{n-1}^2, i-1) = D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = \Phi$, and $D(C_{n-5}^2, i-1) \neq \Phi$.

ii) ($\Rightarrow$) Since $D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = D(C_{n-5}^2, i-1) = \Phi$, by lemma 2.2,

i – 1 $\succ$ n – 2 or i – 1 < $\left\lceil \frac{n-3}{5} \right\rceil$. If i – 1$\prec \left\lceil \frac{n-3}{5} \right\rceil$, then i – 1 $\prec \left\lceil \frac{n-1}{5} \right\rceil$ and hence

$D(C_{n-2}^2, i-1) = \Phi$, a contradiction. So i -1 $\succ$ n – 2. Also, $D(C_{n-1}^2, i-1) \neq \Phi$.

Therefore i $\succ$n-1.Therefore i$\geq$n.But i $\leq$ n. Hence i = n.

iii) ($\Rightarrow$) Since $D(C_{n-1}^2, i-1) = \Phi$, by lemma 2.2, i – 1$\succ$ n – 1 or i – 1 $\prec \left\lceil \frac{n-1}{5} \right\rceil$

If i – 1 $\succ$ n – 1, then i – 1 $\succ$ n – 2 ,by Lemma 2.2,

$D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = D(C_{n-5}^2, i-1) = \Phi$, a contradiction.

Therefore i $\prec \left\lceil \frac{n-1}{5} \right\rceil$ +1. But i –1 $\geq \left\lceil \frac{n-2}{5} \right\rceil$,because $D(C_{n-2}^2, i-1) \neq \Phi$.

Therefore i $\geq \left\lceil \frac{n-2}{5} \right\rceil$+1.

Hence, $\left\lceil \frac{n-2}{5} \right\rceil$+1 $\leq$ i $\prec \left\lceil \frac{n-1}{5} \right\rceil$ +1.

This holds only if n = 5k +2 and i = k+1 = $\left\lceil \frac{5k+2}{5} \right\rceil$,for some k$\in$N.

($\Leftarrow$) If n = 5k+2 and i = $\left\lceil \frac{5k+2}{5} \right\rceil$ for some k$\in$N, then by lemma 2.2,

$D(C_{n-1}^2, i-1) = \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) \neq \Phi$, $D(C_{n-4}^2, i-1) \neq \Phi$ and $D(C_{n-5}^2, i-1) \neq \Phi$.

iv) ($\Rightarrow$) Since $D(C_{n-5}^2, i-1) = \Phi$,by lemma 2.2, i – 1 $\succ$ n – 5 or i – 1 $\prec \left\lceil \frac{n-5}{5} \right\rceil$.

Since $D(C_{n-2}^2, i-1) \neq \Phi$, by lemma 2.2, $\left\lceil \frac{n-2}{5} \right\rceil$+ 1 $\leq$ i $\leq$ n – 1. Therefore i – 1 $\prec \left\lceil \frac{n-5}{5} \right\rceil$ is not possible. Therefore i – 1 $\succ$ n – 5. Therefore i $\succ$n-4.Therefore i$\geq$n-3.

Since $D(C_{n-3}^2, i-1) = \Phi$, i-1$\leq$n-4. Therefore i $\leq$ n-3. Hence i = n – 3.

($\Leftarrow$) if i = n – 3 then by lemma 2.2, $D(C_{n-1}^2, i-1) \neq \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) = \Phi$, $D(C_{n-4}^2, i-1) \neq \Phi$ and $D(C_{n-5}^2, i-1) = \Phi$.

v) ($\Rightarrow$) Since $D(C_{n-1}^2, i-1) \neq \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) \neq \Phi$, $D(C_{n-4}^2, i-1) \neq \Phi$ and $D(C_{n-5}^2, i-1) \neq \Phi$, then by applying lemma 2.2, $\left\lceil \frac{n-1}{5} \right\rceil \leq$ i – 1 $\leq$ n – 1,

$\left\lceil \frac{n-2}{5} \right\rceil \leq$ i – 1 $\leq$ n – 2,$\left\lceil \frac{n-3}{5} \right\rceil \leq$ i – 1 $\leq$ (n – 3), $\left\lceil \frac{n-3}{5} \right\rceil \leq$ i – 1$\leq$ (n – 4) and $\left\lceil \frac{n-3}{5} \right\rceil \leq$ i – 1 $\leq$ n – 5. So$\left\lceil \frac{n-1}{5} \right\rceil \leq$ i – 1 $\leq$ (n – 5) and hence $\left\lceil \frac{n-1}{5} \right\rceil$+ 1$\leq$ i $\leq$ n – 4.

($\Leftarrow$) If $\left\lceil \frac{n-1}{5} \right\rceil$+ 1 $\leq$ i $\leq$ n – 4, then the result follows from lemma 2.2.

Theorem 2.7

For every n $\geq$ 6 and i $\geq \left\lceil \frac{n}{5} \right\rceil$,

i) If $D(C_{n-1}^2, i-1) = D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = \Phi$ and $D(C_{n-5}^2, i-1) \neq \Phi$ then $D(C_n^2, i) = \{$ {1, 6,…………….. n – 4 },{2, 7,………….. n – 3 }, {3, 8,………..… n – 2},{4, 9,…………….. n – 1 },{5, 10,…………….. n } $\}$,

ii) If $D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = D(C_{n-5}^2, i-1) = \Phi$ and $D(C_{n-1}^2, i-1) \neq \Phi$ then $D(C_n^2, i) = \{$ [n] $\}$,

iii) If $D(C_{n-1}^2, i-1) = \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) \neq \Phi$, $D(C_{n-4}^2, i-1) \neq \Phi$ and $D(C_{n-5}^2, i-1) \neq \Phi$ then

$D(C_n^2, i)$ = $\{$ {1, 6,……….. n – 4 },{2, 7,……….. n – 3 },{3, 8,……… … n – 2},

{4, 9……… n – 1 },{5, 10,…………….. n } $\}$, $X_3 \cup \{n\}$.$X_4 \cup \{n-1\}$,

$X_5 \cup \begin{cases} \{n-2\} & if\ 2 \in X_5 \\ \{n-3\} & if\ 2 \notin X_5 \end{cases}$, $X_3 \in D(C_{n-3}^2, i-1)$, $X_4 \in D(C_{n-4}^2, i-1)$, $X_5 \in D(C_{n-5}^2, i-1)$ }…… … (2.1)

iv) If $D(C_{n-3}^2, i-1) = \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-1}^2, i-1) \neq \Phi$, then

$D(C_n^2, i)$ = $\{$ [n] - {x}/x∈[n] $\}$.

v) If $D(C_{n-1}^2, i-1) \neq \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) \neq \Phi$, $D(C_{n-4}^2, i-1) \neq \Phi$, $D(C_{n-5}^2, i-1) \neq \Phi$ then

$D(C_n^2, i)$ = $\{$ $X_1 \cup$ {n}, $X_2 \cup$ {n - 1}, $X_3 \cup$ {n-2}, $X_4 \cup$ {n - 3}, $X_5 \cup$ {n-4}/ $X_1 \in D(C_{n-1}^2, i-1)$,

$X_2 \in D(C_{n-2}^2, i-1)$, $X_3 \in D(C_{n-3}^2, i-1)$, $X_4 \in D(C_{n-4}^2, i-1)$, $X_5 \in D(C_{n-5}^2, i-1)$ $\}$ ………….. ...(2.2)

Proof

i) Since, $D(C_{n-1}^2, i-1) = D(C_{n-2}^2, i-1) = D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = \Phi$ and $D(C_{n-5}^2, i-1) \neq \Phi$ by lemma (2.6) (i) n = 5k, and i=k for some k∈N.

The set $\{$ {1, 6,…………….. n – 4 },{2, 7,………….. n – 3 },{3, 8,………..… n – 2},

{4, 9,…………….. n – 1 },{5, 10,…………….. n } $\}$,

has $\frac{n}{5}$ elements and it covers all vertices. The other sets with cardinality $\frac{n}{5}$

$\{$ {1, 6,…………….. n },{2, 7,………….. n },{3, 8,………..… n },

{4, 9,…………….. n },{5, 10,…………….. n } $\}$,

Clearly, all these are not dominating sets. Therefore $\{$ {1, 6,…………….. n – 4 },
{2, 7,………….. n – 3 },{3, 8,………..… n – 2},
{4, 9,…………….. n – 1 },{5, 10,…………….. n } $\}$,

is the only dominating set of cardinality $\frac{n}{5}$ = k = i.

ii) We have $D(C_{n-2}^2, i-1) =$, $D(C_{n-3}^2, i-1) = D(C_{n-4}^2, i-1) = D(C_{n-5}^2, i-1) = \Phi$ and $D(C_{n-1}^2, i-1) \neq \Phi$. By lemma (2.6) (ii). We have i = n. So, $D(C_n^2, i)$ = $\{$ [n] $\}$.

iii) We have $D(C_{n-1}^2, i-1) = \Phi$, $D(C_{n-2}^2, i-1) \neq \Phi$, $D(C_{n-3}^2, i-1) \neq \Phi$, $D(C_{n-4}^2, i-1) \neq \Phi$ and $D(C_{n-5}^2, i-1) \neq \Phi$.

By lemma (2.6). (iii), n = 5k+2 and i = $\left\lceil \frac{5k+2}{5} \right\rceil$ = k+1 for some k∈N.

We denote the families Y_1 = $\{$ {1, 6,……,5k-4 ,5k+1 },{2, 7,……, 5k-3 ,5k+2 },

{3, 8,……… 5k-2 ,5k+3},{4, 9……5k-1, 5k+4 },{5, 10,…………5k,5k+5 } $\}$ and

Y_2 = { $X_3 \cup \{5k+2\}$.$X_4 \cup \{5k+1\}$, $X_5 \cup \begin{cases} \{5k\} & if\ 2 \in X_5 \\ \{5k-1\} & if\ 2 \notin X_5 \end{cases}$, $X_3 \in D(C^2_{n-3}, i-1)$,

$X_4 \in D(C^2_{n-4}, i-1)$, $X_5 \in D(C^2_{n-5}, i-1)$ }

We shall prove that $D(C^2_{5k+2}, k+1) = Y_1 \cup Y_2$ .Since $D(C^2_{5k}, k)$ = { {1, 6,……,5k-4 ,5k+1 }, {2, 7,……, 5k-3 ,5k+2 },{3, 8,…… 5k-2 ,5k+3},{4, 9……5k-1, 5k+4 },{5, 10,…………5k,5k+5 } },then $Y_1 \subseteq D(C^2_{5k+2}, k+1)$ .Also it is obvious that $Y_2 \subseteq D(C^2_{5k+2}, k+1)$.

Therefore $Y_1 \cup Y_2 \subseteq D(C^2_{5k+2}, k+1)$ ………………………………………………………………………(2.3)

Now let Y $\in D(C^2_{5k+2}, k+1)$. Then 5k + 2, 5k+1 or 5k is in Y. If 5k+2 $\in$Y, then by lemma (3), atleast one vertex labeled 1,2 or 3 and 5k+1, 5k or 5k-1 is in Y. If 5k+1 or 5k is in Y, then

Y – {5k+2} $\in D(C^2_{5k+1}, k)$, a contradiction; because $D(C^2_{5k+1}, k) = \Phi$. Hence 5k – 1 $\in$ Y, 5k $\notin$Y and 5k + 1 $\notin$ Y.

Therefore, Y = X $\cup$ {5k+2} for some X$\in D(C^2_{5k}, k)$ .Hence $Y \in Y_1$. Now suppose that 5k + 1 $\in$Y and 5k+2 $\notin$Y. BY lemma (2.3) at least one vertex labeled 5k,

5k–1or 5k–2 is in Y. If 5k$\in$Y then Y–{5k+1}$\in D(C^2_{5k}, k)$ = { {1, 6,……,5k-4 ,5k+1 },{2, 7,……, 5k-3 ,5k+2 },{3, 8,……… 5k-2 ,5k+3},{4, 9……5k-1, 5k+4 },{5, 10,…………5k,5k+5 } } a contradiction because 5 k $\notin$ X for all X $\in D(C^2_{5k}, k)$.

Therefore 5k – 1, or 5k – 2 is in Y, but 5k$\notin$Y.

Thus Y= X $\cup$ {5k+1} for some of X $\in D(C^2_{5k+1}, k)$. So

$D(C^2_{5k+2}, k+1) \subseteq Y_1 \cup Y_2$ ……………………………………………………………………………... (2.4)

From (2.3) and (2.4),

$D(C^2_n, i)$ = { {1, 6,……….. n – 4 },{2, 7,……….. n – 3 },{3, 8,……… … n – 2}, {4, 9……… n – 1 },{5, 10,……………... n } }, $X_3 \cup \{n\}$.$X_4 \cup \{n-1\}$,

$X_5 \cup \begin{cases} \{n-2\} & if\ 2 \in X_5 \\ \{n-3\} & if\ 2 \notin X_5 \end{cases}$, $X_3 \in D(C^2_{n-3}, i-1)$, $X_4 \in D(C^2_{n-4}, i-1)$, $X_5 \in D(C^2_{n-5}, i-1)$ }

iv) If $D(C^2_{n-3}, i-1) = \Phi$, $D(C^2_{n-2}, i-1) \neq \Phi$, $D(C^2_{n-1}, i-1) \neq \Phi$, by lemma (2.6) (iv) i = n –3.

Therefore $D(P^2_n, i)$ = {[n] - {x} / x$\in$[n]}.

For example $D(P^2_6, 5)$ = {[6] - {x} / x$\in$[6]}.

{ {1,2,3,4,5,6}-{1,2,3,4,5,6} /6$\in$[6]} }.

={ {1,2,3,4,5 }-{{1,2,3,4 ,6}-{{1,2 ,4,5,6}-{{1, 3,4,5,6}-{{2,3,4,5,6} }

v) $D(C^2_{n-1}, i-1) \neq \Phi$, $D(C^2_{n-2}, i-1) \neq \Phi$, $D(C^2_{n-3}, i-1) \neq \Phi$, $D(C^2_{n-4}, i-1) \neq \Phi$, $D(C^2_{n-5}, i-1) \neq \Phi$. Let $X_1 \in D(C^2_{n-1}, i-1)$,so atleast one vertex labeled n-1, n – 2 or n – 3 is in X_1.

If n-1, n-2 or n-3 $\in X_1$, then $X_1 \cup$ {n} $\in D(C^2_n, i)$.

Let $X_2 \in D(C^2_{n-2}, i-1)$, then n – 2 or n – 3 or n – 4 is in X_2.

If n-2, n-3 or n-4 $\in X_2$, then $X_2 \cup$ {n – 1} = $D(C^2_n, i)$. Now let $X_3 \in D(C^2_{n-3}, i-1)$ then n – 3, n – 4 or n – 5 is in X_3.

If n – 3 or n – 4 or n – 5 $\in X_3$,then $X_3 \cup$ {n – 2} = $D(C^2_n, i)$. Now let $X_4 \in D(C^2_{n-4}, i-1)$, then n – 4, or n – 5 or n – 6 is in X_4.

If n-4 $\in X_4$,then $X_4 \cup$ {n-3}$\in D(C^2_n, i)$, for X $\in$ {n, n-1}. If n-5 $\in X_5$,then $X_5 \cup$ {n-4}$\in D(C^2_n, i)$.

Thus we have $\{$ $X_1 \cup \{n\}$, $X_2 \cup \{n-1\}$, $X_3 \cup \{n-2\}$, $X_4 \cup \{n-3\}$, $X_5 \cup \{n-4\}$/ $X_1 \in D(C_{n-1}^2, i-1)$, $X_2 \in D(C_{n-2}^2, i-1)$, $X_3 \in D(C_{n-3}^2, i-1)$, $X_4 \in D(C_{n-4}^2, i-1)$, $X_5 \in D(C_{n-5}^2, i-1)$ $\} \subseteq D(C_n^2, i)$ ....(2.5)

Now suppose that n-1 $\in$ Y, n $\in$ Y then by lemma (3), atleast one vertex labeled n-2, n-3 or in Y. If n-2 $\in$Y then Y=(X_3) $\cup$ {n-2} for some, $X_3 \in D(C_{n-3}^2, i-1)$.

Now suppose then, n-5 $\in$Y & n-4 $\in$ Y, then by lemma (3) one vertex labeled n-6, n-7, in Y.

If n-4$\in$Y, then Y = (X_2)$\cup$ {n-1} for $X_2 \in$, $D(C_{n-2}^2, i-1)$,

So $D(C_n^2, i) \subseteq \{$ $X_1 \cup \{n\}$, $X_2 \cup \{n-1\}$, $X_3 \cup \{n-2\}$, $X_4 \cup \{n-3\}$, $X_5 \cup \{n-4\}$/ $X_1 \in D(C_{n-1}^2, i-1)$, $X_2 \in D(C_{n-2}^2, i-1)$, $X_3 \in D(C_{n-3}^2, i-1)$, $X_4 \in D(C_{n-4}^2, i-1)$, $X_5 \in D(C_{n-5}^2, i-1)$ $\}$ …………..(2.6)

From (2.5) & (2.6)

$D(C_n^2, i) = \{$ $X_1 \cup \{n\}$, $X_2 \cup \{n-1\}$, $X_3 \cup \{n-2\}$, $X_4 \cup \{n-3\}$, $X_5 \cup \{n-4\}$/ $X_1 \in D(C_{n-1}^2, i-1)$, $X_2 \in D(C_{n-2}^2, i-1)$, $X_3 \in D(C_{n-3}^2, i-1)$, $X_4 \in D(C_{n-4}^2, i-1)$, $X_5 \in D(C_{n-5}^2, i-1)$ $\}$.

III. Domination Polynomial Of A Square Cycle (C_n^2)

Let $D(C_n^2, x) = \sum_{i=\lceil \frac{n}{5} \rceil}^{n} d(C_n^2, i) x^i$ be the domination polynomial of a cycle C_n^2. In this section, we study this polynomial.

Theorem 3.1

i) If $D(C_n^2, i)$ is the family of dominating sets with cardinality i of C_n^2, then

$$d(C_n^2, i) = d(C_{n-1}^2, i-1) + d(C_{n-2}^2, i-1) + d(C_{n-3}^2, i-1) + d(C_{n-4}^2, i-1) + d(C_{n-5}^2, i-1)$$

ii) For every n $\geq$ 6,

$D(C_n^2, x) = x\left[D(C_{n-1}^2, x) + D(C_{n-2}^2, x) + D(C_{n-3}^2, x) + D(C_{n-4}^2, x) + D(C_{n-5}^2, x)\right]$ with the initial values

$$D(C_1^2, x) = x,\ D(C_2^2, x) = x^2 + 2x,\ D(C_3^2, x) = x^3 + 3x^2 + 3x\ ,$$

$$D(C_4^2, x) = x^4 + 4x^3 + 6x^2 + 4x,\ D(C_5^2, x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x$$

Proof

i) It follows from theorem 2.7

ii) It follows from part (i) and the definition of the domination polynomial.

Using Theorem 3.1, we obtain $d(C_n^2, i)$ for 1≤n≤15 as shown in table 1.

Table 1: $d(C_n^2, i)$, the number of dominating set of C_n^2 with cardinality i.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n															
1	1														
2	2	1													
3	3	3	1												
4	4	6	4	1											
5	5	10	10	5	1										
6	0	15	20	15	6	1									
7	0	14	35	35	21	7	1								
8	0	12	45	70	56	28	8	1							
9	0	9	57	115	114	82	36	9	1						
10	0	5	34	126	126	196	118	45	10	1					
11	0	0	27	150	328	402	314	163	55	11	1				

12	0	0	18	159	462	723	715	477	218	66	12	1			
13	0	0	10	150	588	1164	1431	1191	695	284	78	13	1		
14	0	0	4	126	678	1698	2567	2614	1885	979	362	91	14	1	
15	0	0	1	93	711	2262	4183	5145	4490	2863	1341	453	105	15	1

In the following Theorem, we obtain some properties of $d(C_n^2, i)$.

Theorem 3.2

The following properties hold for the coefficients of $D(C_n^2, x)$;

i) $d(C_{5n}^2, n) = 5$ for every n $\in$ N.

ii) $d(C_n^2, n) = 1$ for every n $\in$ N

iii) $d(C_n^2, n-1) = n$ for every n$\geq$ 2

iv) $d(C_n^2, n-2) = nC_2$ for every n $\geq$ 3

v) $d(C_n^2, n-3) = nC_3$ for every n $\geq$ 4

vi) $d(C_n^2, n-4) = nC_4$ for every n $\geq$ 5

vii) $d(P_{5n-i}^2, n) = n$ for i = 1, …………….., n = 1,…………….

Proof

i) $D(C_{5n}^2, n)$ = $\{$ {1, 6,……….. n – 4 },{2, 7,………….. n – 3 },{3, 8,………..… n – 2},
{4, 9… n – 1}, {5, 10… n} $\}$, we have $d\,(C_{5n}^2, n) = 5$.

ii) Since $D(C_n^2, i)$ = $\{$ [n] $\}$,we have the result.

iii) Since $D(P_n^2, i)$ = {[n] - {x} /x$\in$[n]}, we have $d(C_n^2, n-1) = n$.

iv) By induction on n

The result is true for n = 3

L.H.S. = $d(C_3^2, 1)$ = 3 (from table)

R.H.S = $\left(\dfrac{3.2}{2}\right)$ = 3

Therefore, the result is time for n = 3

Now suppose that the result is true for all numbers less than 'n' and we prove it for n.

By theorem (3.1),

$$d(C_n^2, n-2) = d(C_{n-1}^2, n-3) + d(C_{n-2}^2, n-3) + d(C_{n-3}^2, n-3) + d(C_{n-4}^2, n-3) + d(C_{n-5}^2, n-3)$$

$$= \frac{(n-1)(n-2)}{2} + (n-2) + 1$$

$$= \frac{n^2 - 3n + 2 + 2n - 4 + 2}{2}$$

$$= \frac{n^2 - n}{2}$$

$$= \frac{n(n-1)}{2}.$$

v) By induction on n

The result is true for n = 4

L.H.S = $d(C_4^2, 1)$ = 4 (from table)

R.H.S = $d(C_4^2, 1)$ = $\dfrac{4.3.2}{6}$

Therefore the result is true for n = 1

Now suppose the result is for all natural numbers less than n.

By theorem 3.1,

$$d(C_n^2, n-3) = d(C_{n-1}^2, n-4) + d(C_{n-2}^2, n-4) + d(C_{n-3}^2, n-4) + d(C_{n-4}^2, n-4) + d(C_{n-5}^2, n-4)$$

$$= \frac{(n-1)(n-2)(n-3)}{6} + \frac{(n-2)(n-3)}{2} + (n-3) + 1$$

$$= \frac{1}{6}\left[(n-1)(n-2)(n-3) + 3(n-2)(n-3) + 6(n-3) + 6\ \right]$$

$$= \frac{1}{6}\left[(n-1)(n-2)(n-3) + 3(n-2)(n-3) + 6(n-3+1)\ \right]$$

$$= \frac{1}{6}\left[(n-1)(n-2)(n-3) + 3(n-2)(n-3) + 6(n-2)\ \right]$$

$$= \frac{1}{6}\left[(n-1)(n-2)(n-3) + 3(n-2)(n-3+2)\ \right].$$

$$= \frac{1}{6}\left[(n-1)(n-2)(n-3) + 3(n-2)(n-1)\ \right]$$

$$= \frac{1}{6}\left[(n-1)(n-2)(n-3+3)\ \right]$$

$$= \frac{1}{6}\left[n(n-1)(n-2)\ \right]$$

vi) By induction on n

Let n = 5

L.H.S = $d(C_5^2, 1)$ = 5 (from table)

R.H.S = $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$

= $\frac{5.4.3.2}{1.2.3.4}$

= 5

Therefore the result is true for n =1

Now suppose that the result is true for all natural numbers less than or equal to n.

$$d(C_n^2, n-4) = d(C_{n-1}^2, n-5) + d(C_{n-2}^2, n-5) + d(C_{n-3}^2, n-5) + d(C_{n-4}^2, n-5) + d(C_{n-5}^2, n-5)$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)}{24} + \frac{(n-2)(n-3)(n-4)}{6} + \frac{(n-3)(n-4)}{2} + (n-4) + 1$$

$$= \frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-4) + 24(n-4) + 24\ \right]$$

$$= \frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-4) + 24(n-4+1)\ \right]$$

$$= \frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-4) + 24(n-3)\ \right]$$

$$= \frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-4+2)\ \right]$$

$$= \frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-2)\ \right]$$

$$= \frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4+3)\ \right]$$

$=\frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4)+4(n-2)(n-3)(n-1)\right]$

$=\frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4+4)\right]$

$=\frac{1}{24}\left[n(n-1)(n-2)(n-3)\right]$

Theorem 3.3

i) $\sum_{i=n}^{5n} d\left(C_i^2, n\right) = 5\sum_{i=2}^{5n-5} d\left(C_i^2, n-1\right)$

ii) For every $j \geq \left\lceil \frac{n}{5} \right\rceil$, $d\left(C_{n+1}^2, j+1\right) - d\left(C_n^2, j+1\right) = d\left(C_n^2, j\right) - d\left(C_{n-5}^2, j\right)$

iii) If $S_n = \sum_{i=\left\lceil \frac{n}{5} \right\rceil}^{n} d\left(C_n^2, j\right)$, then for every $n \geq 6$, $S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5}$ with initial values $S_1=1$, $S_2=3$, $S_3=7$, $S_4=15$ and $S_5 = 31$

Proof

i) Proof by induction on n
First suppose that n =2 then,

$\sum_{i=2}^{10} d(C_i^2, 2) = 75 = 5\sum_{i=2}^{5} d\left(C_i^2, 1\right)$

$\sum_{i=k}^{5k} d(C_i^2, k) = \sum_{i=k}^{5k} d\left(C_{i-1}^2, k-1\right) + \sum_{i=k}^{5k} d\left(C_{i-2}^2, k-1\right) + \sum_{i=k}^{5k} d\left(C_{i-3}^2, k-1\right) + \sum_{i=k}^{5k} d\left(C_{i-4}^2, k-1\right)$

$+\sum_{i=k}^{5k} d\left(C_{i-5}^2, k-1\right)$

$=5\sum_{i=k-1}^{5(k-1)} d\left(C_{i-1}^2, k-2\right) + 5\sum_{i=k-1}^{5(k-1)} d\left(C_{i-2}^2, k-2\right) + 5\sum_{i=k-1}^{5(k-1)} d\left(C_{i-3}^2, k-2\right) + 5\sum_{i=k-1}^{5(k-1)} d\left(C_{i-4}^2, k-2\right)$

$+5\sum_{i=k-1}^{5(k-1)} d\left(C_{i-5}^2, k-2\right)$

$=5\sum_{i=k-1}^{5k-5} d\left(C_i^2, k-1\right)$

We have the result.

ii) By Theorem 3.1, We have

$d(C_{n+1}^2, j+1) - d(C_n^2, j+1) =$

$d(C_n^2, j) + d(C_{n-1}^2, j) + d(C_{n-2}^2, j) + d(C_{n-3}^2, j) + d(C_{n-4}^2, j) + d(C_{n-1}^2, j)$

$d(C_n^2, j) + d(C_{n-1}^2, j) + d(C_{n-2}^2, j) + d(C_{n-3}^2, j) + d(C_{n-4}^2, j) + d(C_{n-1}^2, j)$ _

$d(C_{n-1}^2, j) - d(C_{n-2}^2, j) - d(C_{n-3}^2, j) - d(C_{n-4}^2, j) - d(C_{n-5}^2, j)$

$d(C_{n+1}^2, j+1) - d(C_n^2, j+1) = d(C_n^2, j) - d(C_{n-5}^2, j)$

Therefore we have the result

iii) By theorem 3.1, we have

$$S_n = \sum_{i=\lceil \frac{n}{5} \rceil}^{n} d\left(C_n^2, j\right)$$

$$S_n = \sum_{i=\lceil \frac{n}{5} \rceil}^{n} \left[\ d\left(C_{n-1}^2, j-1\right) + d(C_{n-2}^2, j-1) + d(C_{n-3}^2, j-1) + d(C_{n-4}^2, j-1) + d(C_{n-5}^2, j-1)\ \right]$$

$$= \sum_{i=\lceil \frac{n}{5} \rceil-1}^{n-1} d\left(C_{n-1}^2, j-1\right) + \sum_{i=\lceil \frac{n}{5} \rceil-1}^{n-1} d\left(C_{n-2}^2, j-1\right) + \sum_{i=\lceil \frac{n}{5} \rceil-1}^{n-1} d\left(C_{n-3}^2, j-1\right)$$

$$+ \sum_{i=\lceil \frac{n}{5} \rceil-1}^{n-1} d\left(C_{n-4}^2, j-1\right) + \sum_{i=\lceil \frac{n}{5} \rceil-1}^{n-1} d\left(C_{n-5}^2, j-1\right)$$

$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5}$

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