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On Generalized $\psi-|C,\alpha,\beta,\gamma,\delta|_k$-Summability Factor

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Abstract: *In this paper we have established a theorem on* $\psi-|C,\alpha,\beta,\gamma,\delta|_k$*-summability factor, which gives some new results.*

I. Introduction

A positive sequence (b_n) is said to be almost increasing if there exist a positive sequence (c_n) and two positive constants A and B such that $Ac_n \le b_n \le Bc_n$ (Bari [2]).

A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in BV$ if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.

A positive sequence $X = (X_n)$ is said to be quasi-σ-power increasing sequence if there exist a constant $k = k(\sigma, X) \ge 1$ such that $kn^{\sigma} X_n \ge m^{\sigma} X_m, n \ge m \ge 1$ (Leindler [7]).

Let ψ_n be a sequence of complex numbers. Let $\sum a_n$ be a given infinite series with partial sum (s_n). We denote by $z_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the n$^{\text{th}}$ Cesaro means of order (α,β) with $\alpha+\beta > -1$ of the sequences (s_n) and (na_n) respectively (Borwein [5]).

$$z_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha} A_v^{\alpha} s_v$$

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha} A_v^{\beta} v a_v$$

where $A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \alpha+\beta > -1, A_0^{\alpha+\beta} = 1, A_{-n}^{\alpha+\beta} = 0$ for $n > 0$.

The series $\sum a_n$ is said to be summable $|C,\alpha,\beta|_k, k \ge 1$ and $\alpha+\beta > -1$ (Das [6]) if

$$\sum_{n=1}^{\infty} n^{k-1} \left| z_n^{\alpha,\beta} - z_{n-1}^{\alpha,\beta} \right|^k = \sum_{n=1}^{\infty} \frac{1}{n} \left| t_n^{\alpha,\beta} \right|^k < \infty$$

The series $\sum a_n$ is said to be summable $|C,\alpha,\beta,\gamma,\delta|_k$, $k \ge 1$ $\alpha+\beta > -1$, $\delta \ge 0$ and γ is a real number (Bor [4]) if

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1} \left| z_n^{\alpha,\beta} - z_{n-1}^{\alpha,\beta} \right|^k = \sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k} \left| t_n^{\alpha,\beta} \right|^k < \infty$$

The series $\sum a_n$ is said to be summable $\psi-|C,\alpha|_k, k \ge 1, \alpha > -1$ if (Balci [1])

$$\sum_{n=1}^{\infty} |\psi_n (z_n^{\alpha} - z_{n-1}^{\alpha})|^k = \sum_{n=1}^{\infty} n^{-k} |\psi t_n^{\alpha}|^k < \infty$$

And the series $\sum a_n$ is said to be summable $\psi-|C,\alpha,\beta,\gamma,\delta|_k$ if

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)} \left| \psi_n (z_n^{\alpha,\beta} - z_{n-1}^{\alpha,\beta}) \right|^k = \sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k} \left| t_n^{\alpha,\beta} \psi_n \right|^k < \infty$$

II. Known theorem

Tuncer has proved the following theorem

Theorem 2.1 Let $k \geq 1, 0 \leq \delta < \alpha \leq 1$ and γ be a real number such that $(\alpha + \beta + 1 - \gamma(\delta + 1))k > 1$ and let the sequences (B_n) and (λ_n) such that (B_n) is δ-quasi-monotone with

$$| \Delta\lambda_n | \leq | B_n |, \lambda_n \to 0 \text{as} n \to \infty \quad (1)$$

$$\sum_{n=1}^{\infty} n\delta_n \log n < \infty \text{ and } nB_n \log n \quad (2)$$

is convergent. If the sequence $(W_n^{\alpha,\beta})$ defined by

$$W_n^{\alpha,\beta} = | t_n^{\alpha,\beta} |, \alpha = 1, \beta > -1 \quad (3)$$

$$W_n^{\alpha,\beta} = \max_{1 \leq v \leq n} | t_v^{\alpha,\beta} |, 0 < \alpha < 1, \beta > -1 \quad (4)$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{n^{\gamma(\delta k + k - 1)}}{n^k} (W_n^{\alpha,\beta})^k = O(\log m) \text{as} m \to \infty \quad (5)$$

then $\sum a_n \lambda_n$ is summable $| C, \alpha, \beta, \gamma, \delta |_k$.

III. The main result

The aim of this paper is to generalize Theorem 2.1 to $\psi - | C, \alpha, \beta, \gamma, \delta |_k$ summability. We shall prove the following theorem.

Theorem 3.1 Let ψ_n be the sequence of Complex numbers and let the sequence (B_n) & (λ_n) such that the conditions (1), (2), (3), (4) with

$$\sum_{n=1}^{m} \frac{n^{\gamma(\delta k + k - 1)} | \psi_n W_n^{\alpha,\beta} |^k}{n^k} = O(\log m) \text{as} m \to \infty$$

are satisfied then the series $\sum a_n \lambda_n$ is summable $\psi_n - | C, \alpha, \beta, \gamma, \delta |_k$.

IV. Lemmas

We need the following lemmas for the the proof of our theorem

Lemma 4.1 (Mazhar [9]) Under the condition on (B_n) as taken in the statement of the theorem, we have following

$$n\, B_n \log n = O(1) \quad (6)$$

and

$$\sum_{n=1}^{\infty} n \log n \, | \Delta\, B_n | < \infty \quad (7)$$

Lemma 4.2 (Bor [4]) If $0 < \alpha \leq 1, \beta > -1$ and $1 \leq v \leq n$ then

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \right| \quad (8)$$

V. Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the n th (C, α, β) mean of the sequence $(na_n \lambda_n)$ then we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v .$$

Using Abel's transformation.

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta}\ pa_p$$

$$+ \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta}\ va_v$$

$$|T_n^{\alpha,\beta}| \le \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta}\ pa_p \right|$$

$$+ \frac{|\lambda_n|}{A_n^{\alpha+\beta}} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} va_v \right|$$

$$\le \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} W_v^{\alpha,\beta} |\Delta\lambda_v| + |\lambda_n| W_n^{\alpha,\beta}$$

$$= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} \quad \text{(say)}$$

Since

$$|T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}|^k \le 2^k (|T_{n,1}^{\alpha,\beta}|^k + |T_{n,2}^{\alpha,\beta}|^k)$$

In order to complete the proof of theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k} |T_{n,r}^{\alpha,\beta}, \psi_n|^k < \infty, r = 1,2$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$ we get that

$$\sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} \left| T_{n,1}^{\alpha,\beta} \cdot \psi_n \right|^k$$

$$\le \sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} \left| \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \psi_v A_v^{\alpha} A_v^{\beta} W_v^{\alpha,\beta} \Delta\lambda_v \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{|\psi_v|^k}{n^{(\alpha+\beta-1-\gamma(\delta+1))k}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} |\Delta\lambda_v| (W_v^{\alpha\beta})^k \right\} \left\{ \sum_{v=1}^{n-1} |\Delta\lambda_v| \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \frac{|\psi_v|^k}{n^{(\alpha+\beta+1-\gamma(\delta+1))k}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} |B_v| (W_v^{\alpha,\beta})^k \right\} \left\{ \sum_{v=1}^{n-1} |B_v| \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} |B_v| (W_v^{\alpha,\beta})^k \sum_{n=v+1}^{m+1} \frac{|\psi_v|^k}{n^{(\alpha+\beta+1-\gamma(\delta+1))k}}$$

$$= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} |B_v| (W_v^{\alpha,\beta})^k |\psi_v|^k \int_0^{\infty} \frac{dx}{x^{(\alpha+\beta+1-\gamma(\delta+1))k}}$$

$$= O(1) \sum_{v=1}^{m} |B_v| v^{\gamma(\delta k+k-1)-k+1} (W_v^{\alpha,\beta})^k |\psi_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta(v|B_v|)| \sum_{p=1}^{v} p^{\gamma(\delta k+k-1)-k} (W_v^{\alpha,\beta})^k |\psi_v|^k$$

$$+ O(1) m |B_m| \sum_{v=1}^{m} v^{\gamma(\delta k+k-1)-k} (W_v^{\alpha,\beta})^k |\psi_v|^k$$

$$= O(1) \sum_{v-1}^{m-1} |\Delta(v|B_v|) \log v + O(1) m |B_m| \log m$$

$$= O(1)\sum_{v-1}^{m-1} v \mid \Delta B_v \mid \log v + O(1)\sum_{v=1}^{m-1} \mid B_{v+1} \mid \log v + O(1) m \mid B_m \mid \log m$$
$$= O(1) \text{ as } m \to \infty$$
$$\sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} \left| T_{n,2}^{\alpha,\beta} \psi_n \right|^k = O(1)\sum_{n=1}^{m} \mid \lambda_n \mid n^{\gamma(\delta k+k-1)-k} (W_n^{\alpha,\beta}) \mid \psi_n \mid^k$$
$$= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid \sum_{v=1}^{n} v^{\gamma(\delta k+k-1)-k} (W_v^{\alpha,\beta})^k \mid \psi_n \mid^k$$
$$+ O(1) \mid \lambda_m \mid \sum_{v=1}^{m} v^{\gamma(\delta k+k-1)-k} (W_v^{\alpha,\beta})^k \mid \psi_n \mid^k$$
$$= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid \log n + O(1) \mid \lambda_m \mid \log m$$
$$= O(1)\sum_{n=1}^{m-1} \mid B_n \mid \log n + O(1) \mid \lambda_m \mid \log m$$
$$= O(1) \text{ as } m \to \infty$$

by virtue of the hypothesis of the theorem and lemma 1. This completes the proof of the theorem.

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