IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 7, Issue 5 (Jul. - Aug. 2013), PP 62-67
www.iosrjournals.org

Observations on the transcendental Equation

$$5\sqrt[2]{y^2+2x^2}-\sqrt[3]{X^2+Y^2}=(k^2+1)z^2$$

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Abstract: *The transcendental equation with five unknowns given* $5\sqrt[2]{y^2+2x^2}-\sqrt[3]{X^2+Y^2}=(k^2+1)z^2$ *is analysed for its non-zero distinct integral solutions.Various different patterns of integral solutions are illustrated and some interesting relations between the solutions and special numbers are exhibited.*
Keywords: *Surd,transcendental equation,integral points,figurative numbers.*

Notations:

$S_n = 6n(n-1)+1$ -Star number of rank n.

$OH_n = \frac{1}{3}(n(2n^2+1))$ -Octahedral number of rank n.

$t_{m,n} = n\left[1+\frac{(n-1)(m-2)}{2}\right]$-Polygonal number of rank n with size m.

$P_n^m = \frac{n(n+1)}{6}\left[(m-2)n+(5-m)\right]$ -Pyramidal number of rank n with size m.

$PR_n = n(n+1)$ -Pronic number.

$CP_n^6 = -n^3$ -centered hexagonal pyramidal number of rank n.

$CP_n^9 = \frac{n(3n^2-1)}{2}$ - centered Nonagonal pyramidal number of rank n.

$CP_n^9 = \frac{n(30n^2-24)}{6}$ - centered Triaconagonal pyramidal number of rank n.

$F_{4,n,4} = \frac{n(n+1)^2(n+2)}{6}$ -Four dimensional Figurative number of rank n whose generating polygon is a square.

$F_{4,n,6} = \frac{n^2(n+1)(n+2)}{6}$ -Four dimensional Figurative number of rank n whose generating polygon is a pentagon.

$F_{5,n,3} = \frac{n(n+1)(n+2)(n+3)(n+4)}{5!}$ - Five dimensional Figurative number of rank n whose generating polygon is a triangle.

$F_{6,n,3} = \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{6!}$ - Six dimensional Figurative number of rank n whose generating polygon is a triangle.

I. Introduction

Diophantine equations have an unlimited field of research by reason of their variety. Most of Diophantine problems are algebraic equations [1, 2].It seems that much work has not been done to obtain integral solutions of the transcendental equations. In this context, one may refer [4-15]. This communications analyses a transcendental equation given by $5\sqrt[2]{y^2+2x^2}-\sqrt[3]{X^2+Y^2}=(k^2+1)z^2$.Infinitely many non-zero integer solutions (x,y,X,Y,z) satisfying the above equation are obtained. Various interesting properties among the values of x,y,X,Y,z are presented.

II. Method of Analysis

The transcendental surd equation with five unknowns to be solved for getting non-zero integral solutions is

$$5\sqrt[2]{y^2+2x^2}-\sqrt[3]{X^2+Y^2}=(k^2+1)z^2 \qquad (1)$$

To start with, the substitution of the transformations

$$x=2pq, y=2p^2-q^2, X=p(p^2+q^2), Y=q(p^2+q^2) \qquad (2)$$

in (1) leads to

$$9p^2+4q^2=(k^2+1)z^2 \qquad (3)$$

The above equation (3) is solved through five different methods and thus one can obtain five different sets of solutions to (1)

2.1 Method 1

Take $z=a^2+b^2$ (4)

Using (4) in (3) and applying the method of factorization, define

$$(3p+iq)=(k+i)(a+ib)^2$$

Equating real and imaginary parts of the above equation, we get

$$3p=k(a^2-b^2)-2ab$$

$$2q=(a^2-b^2)+2kab$$

Taking a=6A,b=6B in the above equations and simplifying ,one gets

$$p=12(k(A^2-B^2)-2AB)$$

$$q=18(a^2-b^2)+2kAB)$$

Substituting p,q in (2), it gives the non-zero distinct integral solutions of (1) as

$$x(k,A,B)=2f_1(k,A,B)f_2(k,A,B)$$

$$y(k,A,B)=2f_1^2(k,A,B)-f_2^2(k,A,B)$$

$$X(k,A,B)=f_1(k,A,B)(f_1^2(k,A,B)+f_2^2(k,A,B))$$

$$Y(k,A,B)=f_2(k,A,B)(f_1^2(k,A,B)+f_2^2(k,A,B))$$

$$z(k,A,B)=36(A^2+B^2)$$

where

$$f_1(k,A,B)=12(k(A^2-B^2)-2AB)$$

$$f_2(k,A,B)=18(A^2-B^2+2kAB)$$

A few properties among the solutions are presented for k=1

1)$t_{3,A+1}(x(A,1)-432(6F_{4,A,6}-2CP_A^9-8PR_A+1))=9072P_A^3$

2)$Y(A,A)+101088OH_A.CP_A^6=33696CP_A^6$

3)$9X(A,1)(4t_{3,A}-t_{4,A}-1)=2Y(A,1)(S_A-3t_{4,A}-4)$

4)$21X(2A,A)+2Y(2A,A)=0$

5)$x(A,A)+1728(24F_{4,A,3}-36P_A^3+t_{4,A}+6PR_A)=0$

6)Each of the following is a nasty number[3]

$$(i)\frac{3x(A,B)Y(A,B)}{X(A,B)},\frac{6y(A,B)X^2(A,B)}{2X^2(A,B)-Y^2(A,B)},\frac{6y(A,B)Y^2(A,B)}{2X^2(A,B)-Y^2(A,B)}$$

$$(ii)66(y(A,A)-x(A,A))$$

$$(iii)3(432z^2(A,B)-3x(A,B))$$

2.2 Method 2

Equation (3) can be written as

$$9p^2+4q^2=(k^2+1)z^2.1 \tag{5}$$

Write '1' as

$$1=\frac{((m^2-n^2)+2mni)((m^2-n^2)-2mni)}{(m^2+n^2)^2},(m>n>0) \tag{6}$$

Substituting (4) and (6) in (5) and using the method of factorization, define

$$(3p+i2q)=\frac{1}{(m^2+n^2)}((k+i)(m^2-n^2+2mni)(a+ib)^2)$$

Equating real and imaginary parts of the above equation, we get

$$3p=\frac{1}{(m^2+n^2)}(k(m^2-n^2)(a^2-b^2)-4mnab-2mn(a^2-b^2)-2ab(m^2-n^2)) \tag{7}$$

$$2q=\frac{1}{(m^2+n^2)}(k(m^2-n^2)(a^2-b^2)-4mnab-2k(mn(a^2-b^2)-2ab(m^2-n^2))) \tag{8}$$

Taking $a=6(m^2+n^2)A, b=6(m^2+n^2)B$ in (7),(8) and (4), the values of p,q,z are given by

$$p=12(m^2+n^2)(kF_1(A,B,m,n)-2F_2(A,B,m,n))$$

$$q=18(m^2+n^2)(F_1(A,B,m,n)+2kF_2(A,B,m,n))$$

$$z=36(m^2+n^2)^2(A^2+B^2)$$

where

$$F_1(A,B,m,n)=(m^2-n^2)(A^2-B^2)-4mnAB$$

$$F_2(A,B,m,n)=mn(A^2-B^2)+AB(m^2-n^2)$$

Substituting p,q in (2), the non-zero distinct integral solutions of (1) are given by

$$x(A,B,m,n)=432(m^2-n^2)^2(kF_1(A,B,m,n)-2F_2(A,B,m,n))(F_1(A,B,m,n)+2kF_2(A,B,m,n))$$

$$y(A,B,m,n)=(m^2-n^2)^2(288(kF_1(A,B,m,n)-2F_2(A,B,m,n))^2)-324(F_1(A,B,m,n)+2kF_2(A,B,m,n))^2$$

$$X(A,B,m,n)=12(m^2-n^2)^3(kF_1(A,B,m,n)-2F_2(A,B,m,n))(144(kF_1(A,B,m,n)-2F_2(A,B,m,n))$$
$$+324(F_1(A,B,m,n)+2kF_2(A,B,m,n))$$

$$Y(A,B,m,n)=18(m^2-n^2)^3(F_1(A,B,m,n)+2kF_2(A,B,m,n))(144(kF_1(A,B,m,n)-2F_2(A,B,m,n))$$
$$+324(F_1(A,B,m,n)+2kF_2(A,B,m,n))$$

$$z(A,B,m,n)=36(m^2+n^2)^2(A^2+B^2)$$

2.2.1Properties

1) $\frac{x(A,1,3,2)}{73008}+1428F_{4,A,4}-672P_A^5-570t_{3,A}\equiv 289(\text{mod}\,429)$

2) $3y(2,B,3,1)+112233600F_{4,B,6}+326707200=43200(4502P_B^5+4399t_{4,B})+27417600S_A$

3) $X(A,1,2,1)+Y(A,1,2,1)+228285000=135000(1217520F_{6,A,3}-3470760F_{5,A,3}$
$+866940F_{4,A,6}-199900CP_A^6+487674PR_A+207167T_{4,A})$

4) $84z(A^2,1,2,1)+x(A,1,2,1)=-64800(96P_{A-1}^3-7t_{4,A})$

5) $x(1,1,m+1,1)=1728(2PR_m+1)^2(4F_{4,M,6}-8P_m^5-8t_{3,m}-1)$

6) $3X(A(A+1),A,2m,m)(11CP_A^6-22t_{4,A}t_{3,A})-10Y(A(A+1),A,2m,m)(F_{4,A,6}-t_{4,A})$
$=12(3Y(A(A+1),A,2m,m)-X(A(A+1),A,2m,m)P_A^5$

2.3 Method 3:
In (5), write '1' as

$$1=(i)^n(-i)^n \tag{9}$$

Substituting (9) in (5) and applying the method of factorization, define

$$(3p+i2q)=i^n(k(a^2-b^2)-2ab)+i(2kab+a^2-b^2)$$

Equating real and imaginary parts of the above equation and taking a=6A,b=6B, we get

$$p=12((k(A^2-B^2)-2AB)\cos\frac{n\pi}{2}-2kAB+A^2-B^2)\sin\frac{n\pi}{2})$$

$$q=18((k(A^2-B^2)-2AB)\sin\frac{n\pi}{2}-2kAB+A^2-B^2)\cos\frac{n\pi}{2}$$

Substituting p,q in (2) and (4) the non-zero distinct integral solutions of (1) are found

$$x(k,n,A,B)=432((f_3(k,A,B)\cos\frac{n\pi}{2}-f_4(k,A,B)\sin\frac{n\pi}{2})f_3(k,A,B)\sin\frac{n\pi}{2}+f_4(k,A,B)\cos\frac{n\pi}{2})$$

$$y(k,n,A,B)=288(f_3\cos\frac{n\pi}{2}-f_4\sin\frac{n\pi}{2})^2+324(f_3\sin\frac{n\pi}{2}+f_4\cos\frac{n\pi}{2})^2$$

$$X(k,n,A,B)=12(f_3\cos\frac{n\pi}{2}-f_4\sin\frac{n\pi}{2})^2(44f_3\cos\frac{n\pi}{2}+f_4\sin\frac{n\pi}{2})^2+324(f_3\sin\frac{n\pi}{2}+f_4\cos\frac{n\pi}{2})$$

$$Y(k,n,A,B)=18(f_3\sin\frac{n\pi}{2}+f_4\cos\frac{n\pi}{2})^2(44f_3\cos\frac{n\pi}{2}+f_4\sin\frac{n\pi}{2})^2+324(f_3\sin\frac{n\pi}{2}+f_4\cos\frac{n\pi}{2})$$

$$z(A,B)=36(A^2+B^2)$$

where

$$f_3(k,A,B)=k(A^2-B^2)-2AB$$

$$f_4(k,A,B)=2kAB+A^2-B^2$$

2.3.1 Properties

1) $x(3,1,A,2)+2592(t_{3,A^2}+CP_A^{30})+5184P_{A-1}^3+11664t_{6,A}+20736=1296(25PR_A+18t_{4,A})$

2) $y(3,1,A,1)+2628t_{4,A}^2-44064P_{A-1}^3\equiv-2628(\text{mod}14328)$

3) $x(k,1,A,2A)=432(3k+4)(4k-3)(6F_{4,A,6}-2CP_A^9-PR_A-t_{4,A})$

4) $437(Y^2(1,1,A,2A)-X^2(1,1,A,2A))$ is a perfect square

2.4 Method 4:
Write (3) as

$$9p^2-k^2z^2=z^2-4q^2$$

which can be written as

$$\frac{3p+kz}{z+2q}=\frac{z-2q}{3p-kz}=\frac{a}{b},b\neq0$$

The above is equivalent to the system of double equations

$$3pb-2qa+(kb-a)z=0$$

$$-3pa-2qb+(ka+b)z=0$$

Employing the method of cross-multiplication, the values of p,qand z are obtained.
Substituting p,q in (2) the non-zero integral solutions of (1) are found to be

$$x(k,a,b) = 2g_1(k,a,b)g_2(k,a,b)$$
$$y(k,a,b) = 2g_1^2(a,b) - g_2^2(k,a,b)$$
$$X(k,a,b) = g_1(k,a,b)(g_1^2(k,a,b) + g_2^2(k,a,b))$$
$$Y(k,a,b) = g_2(k,a,b)(g_1^2(k,a,b) + g_2^2(k,a,b))$$
$$z(a,b) = 6(a^2 + b^2)$$

where

$$g_1(a,b) = 4ab - 2k(b^2 - a^2)$$
$$g_2(a,b) = 6kab + 3(b^2 - a^2)$$

2.5 Method 5

Equation (3) can be written as

$$9p^2 - z^2 = k^2z^2 - 4q^2$$

which can be written as

$$\frac{3p+z}{kz-2q} = \frac{kz+2q}{3p-z} = \frac{a}{b}, b \neq 0$$

Proceeding as in method4,the non-zero distinct integral solutions of (1) are given by

$$x(k,a,b) = 2h_1(k,a,b)h_2(k,a,b)$$
$$y(k,a,b) = 2h_1^2(a,b) - h_2^2(k,a,b)$$
$$X(k,a,b) = h_1(k,a,b)(h_1^2(k,a,b) + h_2^2(k,a,b))$$
$$Y(k,a,b) = h_1(k,a,b)(h_1^2(k,a,b) + h_2^2(k,a,b))$$
$$z(a,b) = 6(a^2 + b^2)$$

where

$$h_1(a,b) = 2(a^2 - b^2) + 4kab$$
$$h_2(a,b) = 6ab - 3k(a^2 - b^2)$$

II. Conclusion

In addition to the above solutions, it is observed that the quintuple ($12k\alpha^2$, $\alpha^2(8-9k^2)$, $2\alpha^3(4+9k^2)$, $3k\alpha^3(4+9k^2)$, 6α) satisfies (1).To conclude,one may search for other choices of solutions under consideration and their corresponding properties.

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