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On An Extension Of Absolute Cesaro Summability Factor

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Abstract: *In this paper we have generalised the theorem of Sulaiman which gives some unknown results and known result.s..*
Keywords: *Absolute summability, increasing sequence, Hölder inequality Minkowski inequality and infinite series.*

I. Introduction

A positive sequence (b_n) is called an almost increasing sequence if there exist a positive increasing sequence (c_n) and two positive constants A and B such that (Bari [2])

$$Ac_n \le b_n \le Bc_n \qquad (1.1)$$

A sequence (λ_n) is said to be of bounded variation denoted by $(\lambda_n) \in BV$ if

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty \qquad (1.2)$$

A positive sequence $x = (x_n)$ is said to be a quasi-σ-power increasing seqeuence if there exist a constant $K = K(\sigma, X) \ge 1$ such that $Kn^{\sigma}X_n \ge m^{\sigma}X_n$ holds for all $n \ge m \ge 1$ (Leindler [5]).

Let (ψ_n) be a sequence of complex numbers and let Σa_n be a given infinite series with partial sums (s_n). We denote by z_n^{α} and t_n^{α} the nth cesaro means of order α with $\alpha > -1$ of the sequence (s_n) and (na_n) respectively, that is

$$z_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \qquad (1.3)$$

$$t_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v \qquad (1.4)$$

where $A_n^{\alpha} = \binom{n+\alpha}{n} = \dfrac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!} = O(n^{\alpha}),\ A_{-n}^{\alpha} = 0$ for $n > 0$

The series Σa_n is said to be summable $\psi - |C,\alpha|_k,\ k \ge 1$ and $\alpha > -1$ if (Balci [1])

$$\sum_{n=1}^{\infty} |\psi_n (z_n^{\alpha} - z_{n-1}^{\alpha})|^k = \sum_{n=1}^{\infty} n^{-k} |\psi_n t_n^{\alpha}|^k < \infty \qquad (1.5)$$

If $\psi_n = n^{1-\frac{1}{k}}$ then $\psi - |C,\alpha|_k$-summability is the same as $|C,\alpha|_k$-summability (Flett [4]).

II. Known theorem

Sulaiman [6] has proved the following theorem.

Theorem 2.1 Let (ψ_n) be a sequence of positive real numbers. Let (X_n) be a quasi-f-increasing sequence $f = (f_n), f_n = n^{\beta}(\log n)^{\gamma}$, $0 < \beta \le 1$, $\gamma \ge 0$. Let (λ_n) and (μ_n) be sequences of numbers such that (μ_n) is positive non-decreasing sequences.

If

$$\sum_{n=v}^{m} \frac{\psi_n^{k-1}}{n^{k+1}} = O\left(\frac{\psi_v^{k-1}}{v^k}\right) \tag{2.1}$$

$$\sum_{n=1}^{\infty} n^{\beta+1} (\log n)^{\gamma} X_n \, |\Delta^2 \lambda_n| < \infty \tag{2.2}$$

$\lambda_n \to 0$ as $n \to \infty$ (2.3)

$n^{1+\beta} (\log n)^{\gamma} X_n \mu_n \Delta\left(\frac{1}{\mu_n}\right) = O(1)$ as $n \to \infty$ (2.4)

$$\sum_{n=2}^{m} \frac{\psi_n^{k-1} |t_n|^k}{n^k (n^{\beta} (\log n)^{\gamma} X_n)^{k-1}} = O(m^{\beta} (\log m)^{\gamma} X_m \mu_m) \text{ an } m \to \infty \tag{2.5}$$

$\sum_{n=1}^{m} \frac{|\lambda_n|}{n} < \infty$ (2.6)

and

$$\mu_n \Delta^2 \left(\frac{1}{\mu_n}\right) = O\left(\frac{|\Delta \lambda_n|}{n |\lambda_{n+1}|}\right) \tag{2.7}$$

are satisfied then the series $\Sigma a_n \lambda_n \mu_n$ is summable $\psi - (C,1)_k$, $k \geq 1$.

III. Main theorem

In this paper we have proved the following theorem.

Theorem 3.1 Let (ψ_n) be a sequence of complex numbers. Let (X_n) be a quasi-f-power increasing sequence, $f = (f_n)$, $f_n = n^{\beta} (\log n)^{\gamma}, 0 < \beta \leq 1, \gamma \geq 0$. Let (λ_m) and (μ_n) be sequences of the numbers such that (μ_n) is positive non-decreasing sequence if (2.1), (2.2), (2.3) (2.4), (2.6) (2.7) and

$$\sum_{n=2}^{m} \frac{\psi_n^{k} |W_n^{\alpha}|^k}{n^k (n^{\beta} (\log n)^{\gamma} X_n)^{k-1}} = O((m^{\beta} (\log m)^{\gamma} X_m)^k \mu_m)$$

where $W_n^{\alpha} = \begin{cases} |t_n^{\alpha}|, & \alpha = 1 \\ \max 1 \leq v \leq n |\mathrm{t}_v^{\alpha}|, & 0 < \alpha < 1 \end{cases}$

Are satisfied then $\sum \frac{a_n \lambda_n}{\mu_n}$ is summable $\psi - |C, \alpha|_k, k \geq 1$.

IV. Lemmas

We have need the following lemmas for the the proof of our theorem.

Lemma 4.1 (Sulaiman [6]) Let (X_n) be a positive non decreasing sequence and, let (λ_n) be a sequence of numbers if

$$\lambda_m = O(1) \ , \quad m \to \infty$$

$$|\lambda_n| X_n = O(1) \ , \quad n \to \infty$$

$$\sum_{n=1}^{m} n |\Delta^2 \lambda_n| X_n = O(1) \quad m \to \infty$$

are satisfied then

neither $\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} < \infty$, nor $\sum_{n=1}^{\infty} |\lambda_n| < \infty$

Lemma 4.2 (Sulaiman [6]) Let (X_n) be a quasi-f-power increasing sequence

$f=(f_n), f_n=n^{\beta}(\log n)^{\gamma},\ 0\le\beta\le1,\ \gamma\ge0.$ if

$\lambda_n\to0$ as $n\to\infty$

$$\sum_{n=1}^{\infty}n^{\beta+1}(\log n)^{\gamma}X_n\,|\Delta^2\lambda_n|<\infty$$

Then $m^{\beta+1}(\log m)^{\gamma}X_n\,|\Delta\lambda_m|=O(1)$ as $m\to\infty$

$$\sum_{n=1}^{\infty}n^{\beta}(\log n)^{\gamma}X_n\,|\Delta\lambda_n|=O(1)$$

and $n^{\beta}(\log n)^{\gamma}X_n\,|\lambda_n|=O(1)$ as $n\to\infty$.

V. Proof of the theorem

Let (T_n^{α}) be the n^th^ (C,α) with $0<\alpha\le1$, mean of the sequ ence $\left(\frac{na_n\lambda_n}{\mu_n}\right)$, then

$$T_n^{\alpha}=\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}\frac{va_v\lambda_v}{\mu_v}$$

Applying Abel's transformation and using lemma (4.1), we get that

$$T_n^{\alpha}=\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1}\Delta\left(\frac{\lambda_v}{\mu_v}\right)\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p+\frac{\lambda_n}{\mu_nA_n^{\alpha}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v$$

$$|T_n^{\alpha}|\le\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1}\left|\Delta\left(\frac{\lambda_v}{\mu_v}\right)\right|\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p\right|+\left|\frac{\lambda_n}{\mu_n}\right|\frac{1}{A_n^{\alpha}}\left|\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\right|$$

$$\le\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1}A_v^{\alpha}W_v^{\alpha}\left|\frac{\Delta\lambda_v}{\mu_v}\right|+\left|\frac{\lambda_n}{\mu_n}\right|W_n^{\alpha}$$

$$=\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1}W_v^{\alpha}A_v^{\alpha}\left\{\left(\Delta\left(\frac{1}{\mu_v}\right)\lambda_v+\left(\frac{\Delta\lambda_v}{\mu_{v+1}}\right)\right)\right\}+\left|\frac{\lambda_n}{\mu_n}\right|W_n^{\alpha}$$

$$=T_{n,1}^{\alpha}+T_{n,2}^{\alpha}+T_{n,3}^{\alpha}$$

To complete the proof of the theorem by minkowskiys inequality, it is sufficient to show that

$$\sum_{k=1}^{\infty}n^{-k}\,|\psi_nT_{n,r}^{\alpha}|^k<\infty\ \text{ for }\ r=1,2,3$$

Now, when $k>1$ applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'1}=1$ we get that

$$\sum_{n=2}^{m+1}n^{-k}\,|\psi_nT_{n,1}^{\alpha}|^k\le\sum_{n=2}^{m+1}n^{-k}(A_n^{\alpha})^{-k}\,|\psi_n|^k\left\{\sum_{v=1}^{n-1}A_v^{\alpha}W_v^{\alpha}\,|\lambda_v|\,\Delta\left|\frac{1}{\mu_v}\right|\right\}^k$$

$$\le\sum_{n=2}^{m}\frac{|\psi_n|^k}{n^k}(n)^{-\alpha k}\sum_{v=1}^{n-1}|W_v^{\alpha}|^k\,v^{\alpha k}\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|^k\left(\sum_{v=1}^{n-1}\Delta\left|\frac{1}{\mu_v}\right|\right)^{k-1}$$

$$=O(1)\sum_{n=2}^{m}\frac{\psi_n^k}{n^{k(1+\alpha)}}\sum_{v=1}^{n-1}v^{\alpha k}\,|W_v^{\alpha}|^k\,\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|^k$$

$$=O(1)\sum_{v=1}^{m}v^{\alpha k}\,|W_v^{\alpha}|^k\,\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|^k\sum_{n=v}^{m}\frac{\psi_n^k}{n^{k(1+\alpha)}}$$

$$= O(1)\sum_{v=1}^{m}\frac{v^{\alpha k}\,|W_v^{\alpha}|^k}{v^{k+\alpha k-1}(v^{\beta}(\log v)^{\gamma}X_v)^{k-1}}\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|\psi_v^k(|\lambda_v|v^{\beta}(\log v)^{\gamma}X_v|^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\frac{v\,|W_v^{\alpha}|^k}{v^{k}(v^{\beta}(\log v)^{\gamma}X_v)^{k-1}}\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|\psi_v^k$$

$$= O(1)\sum_{v=1}^{m}\frac{\psi_v^k\,|W_v^{\alpha}|^k}{v^{k}(v^{\beta}(\log v)^{\gamma}X_v)^{k-1}}\Delta|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)$$

$$= O(1)\sum_{n=2}^{m}\frac{\psi_v^k}{n^{k(1+\alpha)}}\sum_{v=1}^{n-1}v^{\alpha k}|W_v^{\alpha}|^k\ \Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|^k$$

$$= O(1)\sum_{v=1}^{m}\left(\sum_{r=1}^{v}\frac{\psi_r^k\,|W_r^{\alpha}|^k}{r^{k}(r^{\beta}(\log r)^{\gamma}X_r)^{k-1}}\Delta(v|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)\right)$$

$$+O(1)\left(\sum_{v=1}^{m}\frac{\psi_v^k\,|W_v^{k}|^k}{v^{k}(v^{\beta}(\log v)^{r}X_v)^{k-1}}\right)m(\lambda_m)\Delta\left(\frac{1}{\mu_m}\right)$$

$$= O(1)\sum_{v=1}^{m}v^{\beta}(\log v)^{\gamma}X_v\mu_v\left(|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)+(v+1)|\Delta\lambda_v|\Delta\left|\frac{1}{\mu_v}\right|+(v+1)|\lambda_{v+1}|\Delta^2\left|\frac{1}{\mu_v}\right|\right)$$

$$+O(1)m^{\beta+1}(\log m)^{\gamma}X_m\mu_m|\lambda_m|\Delta\left(\frac{1}{\mu_m}\right)$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_v|}{v}+O(1)\sum_{v=1}^{m-1}v^{\beta}(\log v)^{\gamma}X_v|\Delta\lambda_v|$$

$$+O(1)\sum_{v=1}^{m-1}\gamma^{\beta}(\log v)^{\gamma}X_v|\Delta\lambda_v|+O(1)$$

$$= O(1)$$

$$\sum_{n=2}^{m}\frac{|\psi_n T_{n,2}^{\alpha}|^k}{n^k}=\sum_{n=2}^{m}\frac{\psi_n^k}{n^k}\left|\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1}A_v^{\alpha}W_v^{\alpha}\ \frac{\Delta\lambda_v}{\mu_{v+1}}\right|^k$$

$$= O(1)\sum_{n=2}^{m}\frac{\psi_n^k}{n^{k+\alpha k}}\sum_{v=1}^{n-1}\frac{v^{\alpha k}|W_v^{\alpha}|^k}{(v^{\beta}(\log v)^{\gamma}X_v)^{k-1}}\frac{\Delta\lambda_v}{\mu_{v+1}^k}\left(\sum_{v=1}^{n-1}v^{\beta}(\log v)^{\gamma}X_v\Delta\lambda_v\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\frac{v^{\alpha k}|W_v^{\alpha}|^k}{(v^{\beta}(\log v)^{\gamma}X_v)^{k-1}}\frac{\Delta\lambda_v}{\mu_{v+1}^k}\sum_{n=v}^{m}\frac{\psi_n^k}{n^{k+\alpha k}}$$

$$= O(1)\sum_{v=1}^{m}\frac{v|W_v^{\alpha}|^k}{(v^{\beta}(\log v)^{\gamma}X_v)^{k-1}}\frac{\Delta\lambda_v}{\mu_{v+1}^k}\sum_{n=v}^{m}\frac{\psi_n^k}{n^{k+1}}$$

$$= O(1)\sum_{v=1}^{m}\frac{|W_v^{\alpha}|^k\,\psi_n^k}{v^k(v^{\beta}(\log v)^{\gamma}X_v)^{k-1}}\frac{v|\Delta\lambda_v|}{\mu_v}$$

$$= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{|W_r^{\alpha}|^k\,\psi_r^k}{r^k(r^{\beta}(\log r)^{\gamma}X_r)^{k-1}}\right)\Delta\left(\frac{v|\Delta\lambda_v|}{\mu_v}\right)$$

$$+\left(\sum_{v=1}^{m}\frac{|W_v^{\alpha}|^k\,\psi_v^k}{v^k(\mathrm{v}^{\beta}(\mathrm{logv})^{\gamma}X_v)^{k-1}}\right)\frac{m|\Delta\lambda_m|}{\mu_m}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} v^{\beta}(\log v)^{\gamma} X_v \mu_v \left(\Delta\left(\frac{1}{\mu_v}\right)\left(v\,|\Delta\lambda_v| + \frac{1}{\mu_{v+1}}\left(|\Delta\lambda_v| + (v+1)\,|\Delta^2\lambda_v|\right)\right)\right) \\
&+ O(1) m X_m |\Delta\lambda_m| \\
&= O(1)\sum_{v=1}^{m-1} v^{\beta}(\log v)^{\gamma} X_v |\Delta\lambda_v| + O(1)\sum_{v=1}^{m-1} v^{\beta}(\log v)^{\gamma} X_v |\Delta\lambda_v| \\
&+ O(1)\sum_{v=1}^{m-1} v^{\beta+1}(\log v)^{\gamma} X_v |\Delta^2\lambda_v| + O(1) \\
&= O(1)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{m} \frac{|\psi_n T_{n,3}^{\alpha}|^k}{n^k} &= O(1)\sum_{n=1}^{m} \frac{\psi_n^k}{n^k}\left|\frac{W_n^{\alpha}\lambda_n}{\mu_n}\right|^k \\
&= O(1)\sum_{n=1}^{m} \frac{\psi_n^k |W_n^{\alpha}|^k}{n^k (n^{\beta}(\log n)^{\gamma} X_n)^{k-1}} \frac{|\lambda_n|}{\mu_n^k} (n^{\beta}(\log n)^{\gamma} X_n |\lambda_n|)^{k-1} \\
&= O(1)\sum_{n=1}^{m} \frac{\psi_n^k |W_n^{\alpha}|^k}{n^k (n^{\beta}(\log n)^{\gamma} X_n)^{k-1}} \frac{|\lambda_n|}{\mu_n} \\
&= O(1)\sum_{n=1}^{m-1}\left(\sum_{v=1}^{n} \frac{|W_v^{\alpha}|^k \psi_v^k}{v^k (v^{\beta}(\log v)^{\gamma} X_v)^{k-1}} \right)\Delta\left(\frac{v|\Delta\lambda_n|}{\mu_n}\right) \\
&+ \left(\sum_{n=1}^{m} \frac{\psi_n^k |W_n^{\alpha}|^k}{n^k (n^{\beta}(\log n)^{\gamma} X_n)^{k-1}} \right)\frac{|\lambda_m|}{\mu_m} \\
&= O(1)\sum_{n=1}^{m-1} n^{\beta}(\log n)^{\gamma} X_n \mu_n \left(\Delta\left(\frac{1}{\mu_n}\right)|\lambda_n| + \frac{|\Delta\lambda_n|}{\mu_{n+1}} \right) + O(1) X_m |\lambda_m| \\
&= O(1)\sum_{n=1}^{m-1} \frac{|\lambda_n|}{n} + O(1)\sum_{n=1}^{m-1} n^{\beta}(\log n)^{\gamma} X_n |\Delta\lambda_n| + O(1) \\
&= O(1)
\end{aligned}
$$

This completes the proof of theorem.

References

[1] Balci, M.; Absolute ψ -summability factors, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 29, 1980.
[2] Bari, N.K.; Best approximation and differential properties of two conjugate functions, T. Mat. Obs. 5, 1956.
[3] Bor, H.; A new application of quasi power-increasing sequences II, Bor fixed point theory and application, 2013.
[4] Flett, T.M.; On an extension of absoulte summability and some theorems of littlewood and paley, Proc. London Math. Soc. 7, 1957.
[5] Leindler, L.; A new application of quasi power increasing sequences, Pub. Math. (Debar.) 58, 2001.
[6] Sulaiman, W.T.; On some generalization of absolute Cesaro summability factors, J. of classical Analysis Vol. 1, 2012.