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Existence of Solutions for Nonlinear Neutral Integrodifferential Equations with Infinite Delay.

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Abstract*: In the present paper, we investigate the existence of solutions of neutral integrodifferential equations with infinite delay. The results are obtained by using Schaefer's fixed point theorem and rely on a priori bounds of solution and the inequality established by Pachpatte.*

Keywords: *Existence: Nonlinear Neutral integrodifferential equations: Schaefer's fixed point theorem: Pachpatte's inequality.*

I. Introduction

In the last few years several papers have been devoted to the study of existence and uniqueness of solutions of nonlinear differential equations with nonlocal conditions. Among others, we refer to the papers of Balachandran and Chandrasekarn [3], Balachandran and llamaran [2], Ntouyas and Tsamatos [7]. Ferenc [6] proved an existence theorem for Volterra integrodifferential equations with infinite delays using Schauder's fixed point theorem.

Neutral differential equations arise in many areas of applied Mathematics and such equations have received much attention in recent years. Eke and Jackreece [5] studied the existence results of nonlinear neutral integrodifferential system with distributed delays using Schaefer's theorem. Balachandran and Rajagounder [4] studied the existence of mild solutions of nonlinear integrodifferential equations with time varying delays in Banach Spaces.

Pazy [8] used the method of semigroups to study the existence and uniqueness of mild, strong and classical solutions of semilinear evolution equation. In Sathiyanathan and Nandha [11] sufficient condition for the existence of mild solutions of a class of nonlinear evolution integrodifferential equation in Banach Spaces was proved using resolvent operator and Schaefer's fixed point theorem.

Rupali and Dhakne [10] investigate the global existence of solution of nonlinear integrodifferential equation by using the topological transversality theorem of Leray-Schauder's theorem and a priori bounds of solution established by Pachpatte. Adderrazzak et al [1] used semigroup theory and Banach fixed point theory to establish existence and uniqueness of integral solutions for a class of neutral Volterra integrodifferential equations with infinite delay. The purpose of this paper is to study the existence of mild solutions for nonlinear neutral integrodifferential equations with infinite delay with the help of Schaefer's fixed point theorem.

II. Preliminaries

Consider the neutral functional integrodifferential equations with infinite delay of the form

$$\frac{d}{dt}\left[x(t)-h(t,x_t)\right]=Ax(t)+Bu(t)+\int_{-\infty}^{t}g(t,s,x_s)ds+f(t,x_t), \quad t\ge 0$$
$$x_0=\phi\in\Re \tag{2.1}$$

where the state $x(.)$ takes values in the Banach space X endowed with the norm $|.|$, the control function $u(.)$ is given in the Banach space of admissible control function $L^2(J,U)$ with U as a Banach space. $A:D(A)\to X$ is an infinitesimal generator of a strongly continuous semigroup of bounded linear operator $T(t),\ t\ge 0$ in X. $\Re$ is a bounded linear operator from U into X, where $g:J\times J\times\Re\to X$, $f:J\times\Re\to X$, and $h:J\times\Re\to X$ are given functions. The delay $x_t:(-\infty,0]\to X$ defined by $x_t(\theta)=x(t+\theta)$ belongs to some abstract phase space $\Re$, which will be a linear space of functions mapping $(-\infty,0]$ into X endowed with the seminorm $\|.\|_{\Re}$ in $\Re$. Throughout this paper we assume that $\Re$ satisfies the following axioms

(A1) If $x:(-\infty,a)\to X, a>0,$ is continuous on $[0,a]$ and $x_0\in\Re$, then for every $t\in[0,a]$ the following conditions hold:

(i) x_t is in $\Re$

(ii) $\|x(t)\| \le H\|x_t\|_{\Re}$

(iii) $\|x_t\|_{\Re} \le K(t)\sup\{|x(s)| : 0 \le s \le t\} + M(t)\|x_0\|_{\Re}$

where $H \ge 0$ is a constant; $K, M : [0,\infty) \to [0,\infty)$, K is continuous and M is locally bounded, and H, K, M are independent of $x(.)$.

(A2) For the function $x(.)$ in (A1), x_t is a $\Re$-valued continuous function on $[0, a]$.

(A3) The space $\Re$ is complete.

Definition 2.1. A continuous function $x(t)$ is said to be a unique mild solution of the system (2.1) for each $u \in L^2(J,u)$ as given by [ ] if

$$x(t) = T(t)[\phi(0) - h(0,\phi)] + h(t, x_t) + \int_0^t AT(t-s)h(s, x_t)ds$$
$$+ \int_0^t T(t-s)\left[Bu(s) + f(s, x_s) + q(s) + \int_0^t g(s,\tau, x_s)d\tau\right]ds \quad (2.2)$$

is satisfied.

We need the following fixed point due to Schaefer [12]

Theorem 2.1 (Schaefer). Let E be a normed linear space. Let $\Gamma : E \to E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let

$$\xi(\Gamma) = \{x \in E : x = \lambda \Gamma x \ \ for\ \ some\ 0 < \lambda < 1\}$$

Then either $\xi(\Gamma)$ is unbounded or Γ has a fixed point.

Theorem 2.2 (Pachpatte [9]). Let $a(t), b(t)\ and\ c(t)$ be nonnegative continues functions defined on R^+, for which the inequality

$$c(t) \le c_0 + \int_0^t a(t)c(t)ds + \int_0^t a(s)\left[\int_0^s b(\tau)c(\tau)d\tau\right]ds$$

Holds for all $t \in R^+$, where c_0 is a nonnegative constant. Then

$$c(t) \le c_0\left[1 + \int_0^t a(s)\exp\left(\int_0^t (a(\tau) + b(\tau))d\tau\right)ds\right]$$

For all $t \in R^+$.

We assume the following hypothesis.

(H1) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t),\ t > 0$ on X and their exists $M \le 1$ and $M_1 > 0$ such that

$$|T(t)| \le M \ \text{ and } \ |AT(t)| \le M_1.$$

(H2) For $h : J \times \Re \to X$ is continuous and there exists constants $L', L \ge 0$ such that

$$\|h(t,\psi_1) - h(s,\psi_2)\| \le L'[|t-s| + \|\psi_1 - \psi_2\|]$$

For every $0 \le s \le t \le b$, $\psi_1, \psi_2 \in \Re$, and the inequality $\|h(t,\psi)\| \le L$ holds for $t \in [0,a], \psi \in \Re$.

(H3) $f : J \times \Re \to X$ is continuous and there exists $M_2 > 0$, such that

$$\|f(t,\psi)\| \le M_2$$

for every $0 \le s \le t \le a\ \ and\ \ \psi \in \Re$.

(H4) $g : J \times J \times \Re \to X$ is continuous and there exists $N > 0$, such that

$$\|g(t,s,\eta)\| \le N$$

For every $0 \le s \le t \le a.$

(H5) for each $\phi \in \Re$, $q(t) = \lim_{a\to\infty} \int_{-a}^{0} g(t,s,\phi(s))ds$ exists and it is continuous. Further there exists $N_1 > 0$ such that $\|q(t)\| \le N_1$.

(H6) there exists a compact set $V \subseteq X$, such that $T(t)f(s,\psi), T(t)Bu(s), T(t)g(s,\tau,\eta)$ and $T(t)q(s) \in V$ for all $\psi,\eta \in \Re$, and $0 \le \tau \le s \le t \le a.$

(H7) The linear operator $W : L^2(J,U) \to X$ is defined by

$$Wu = \int_0^b T(b-s)Bu(s)ds$$

is a solution map and there exists a constant $M_3 > 0$ such that $|B| \le M_3$.

1. MEAN RESULT

Theorem 3.1. Assume that hypothesis H1 – H7 holds, then there exists a unique mild solution of (2.1) such that $x(\phi : u) \in X$ for each $u \in L^2(J,U)$, $\phi \in \Re$.

From theorem 3.1 we define the solution mapping $W : L^2(J,u) \to X$ given by

$$(Wu)(t) = x_t(\phi : u).$$

Theorem 3.2. Let $u \in U$ and $\phi \in \Re$, then under the hypothesis H1- H7 the solution map

$$(Wu)(t) = x_t(\phi : u)$$

satisfies

$$\|x_t(\phi : u)\|_X \le c\left(M\|\phi\|_{\Re} + M_B\|u\|_{L^2(J.U)}{}^{\sqrt{T}}\right)$$

Where c depends on $M, M_1, M_2, L, N \, and \, N_1$.

Proof

First we establish the priori bounds on the solution map. From hypothesis, we have

$$\|x_t(\phi : u)(\theta)\| \le \|T(t+\theta)[\phi(0) - h(0,\phi)]\| + \|h(t,x_t)\| + \int_0^{t+\theta} \|AT(t+\theta-s)h(s,x_s)\|ds$$
$$+ \int_0^{t+\theta} \left\| T(t+\theta-s)\left[Bu(s) + f(s,x_s) + q(s) + \int_0^s g(s,\tau,x_\tau)d\tau \right]\right\| ds$$
$$\le M\|\phi\| + ML\|\phi\| + L\|x_t\| + M_B\|u\| + MN_1$$
$$+ (M_1L + MM_2)\int_0^{t+\theta} \|x_s\|ds + M\int_0^{t+\theta} N\left(\int_0^s \|x_\tau\|d\tau\right)ds$$

where

$$M_B = M\|B\|$$

Hence

$$\|x_t(\phi : u)\| = \sup\|x_t(\phi : u)(\theta)\|$$
$$\le \frac{L^*}{1-L}\left(M\|\phi\| + M_B\|\phi\|\right) + \frac{M_1L + MM_2}{1-L}\int_0^t \|x_s\|ds$$
$$+ \frac{M}{1-L}\int_0^t N\left(\int_0^s \|x \ \|d\tau\right)ds$$

and

$$L^* = L + 1$$

By the above result, we have

$$\|x_t(\phi:u)\| \le \frac{L^*}{1-L}\left(M\|\phi\| + M_B\|u\|\right) + \frac{M_1 L + MM_2}{1-L}\int_0^t \|x_s\| ds$$
$$+ \frac{M}{1-L}\int_0^t N\left(\int_0^s \|x_\tau\| d\tau\right) ds$$

From theorem 2.2.

$$\|x_t(\phi:u)\| \le \frac{L^*}{1-L}\left(M\|\phi\| + M_B\|u\|\right)$$
$$\times\left[1 + \int_0^t \frac{M_1 L + MM_2}{1-L}\|x_s\| \exp\left(\int_0^s \left(\frac{MN}{1-L} + \|x_\tau\|\right) d\tau\right) ds\right]$$

Therefore,

$$\|x_t(\phi:u)\| \le c\left(M\|\phi\| + M_B\|u\|\right)$$

We now rewrite the initial value problem (2.1) as follows:

For $\phi \in \Re$, define $\hat{\phi} \in \Re$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t) - h(t,\phi) & if \ -\infty \le t \le 0 \\ T(t)[\phi(0) - h(0,\phi)] & if \ 0 \le t \le a \end{cases}$$

If $y \in \Re$ *and* $x(t) = y(t) + \hat{\phi}(t),\ t \in [-\infty, a]$, then it is easy to see that x satisfies (2.2) if and only if y satisfies,

$$y(t) = y_0 = 0, \quad -\infty \le t \le 0$$

$$y(t) = \int_0^t AT(t-s)h\left(s, y_s + \hat{\phi}_s\right) ds$$
$$+ \int_0^t T(t-s)\left[Bu(s) + f\left(s, y_s + \hat{\phi}_s\right) + q(s) + \int_0^t g\left(s,\tau, y_\tau + \hat{\phi}_s\right) d\tau\right] ds \qquad t > 0$$

We define the operator $\Gamma : \Re_0 \to \Re_0,\ \Re_0 = \{y \in \Re : y_0 = 0\}$ by

$$(\Gamma y)(t) = \begin{cases} 0 \\ \int_0^t AT(t-s)h\left(s, y_s + \hat{\phi}_s\right) ds \\ + \int_0^t T(t-s)\left[Bu(s) + f\left(s, y_s + \hat{\phi}_s\right) + q(s) + \int_0^t g\left(s,\tau, y_\tau + \hat{\phi}_\tau\right) d\tau\right] ds \end{cases} \tag{3.1}$$

From the definition of an operator Γ defined on equation (3.1), it can be noted that the equation (2.2) can be written as

$$y(t) = \lambda \Gamma y(t), \quad 0 < \lambda < 1 \tag{3.2}$$

Now, we prove that Γ is completely continuous. For any $y \in B_k$, let $0 < t_1 < t_2 < a$, then

$$\|(\Gamma y)(t_1) - (\Gamma y)(t_2)\| \le \|T(t_1) - T(t_2)\| \|\phi(0) - h(0,\phi)\| + \left\|h\left(t_1, y_{t_1} + \hat{\phi}_{t_1}\right) - h\left(t_2, y_{t_2} + \hat{\phi}_{t_2}\right)\right\|$$
$$+ \left\|\int_0^{t_1} A[T(t_1 - s) - T(t_2 - s)]h\left(s, y_s + \hat{\phi}_s\right) ds\right\| + \left\|\int_{t_1}^{t_2} AT(t_2 - s)h\left(s, y_s + \hat{\phi}_s\right) ds\right\|$$
$$+ \left\|\int_0^{t_1} [T(t_1 - s) - T(t_2 - s)]\left[Bu(s) + f\left(s, y_s + \hat{\phi}_s\right) + q(s) + \int_0^s g\left(s,\tau, y_\tau + \hat{\phi}_\tau\right) d\tau\right] ds\right\|$$
$$+ \left\|\int_{t_2}^{t_1} T(t_2 - s)\left[Bu(s) + f\left(s, y_s + \hat{\phi}_s\right) + q(s) + \int_0^s g\left(s,\tau, y_\tau + \hat{\phi}_s\right) d\tau\right] ds\right\|$$
$$\le \|T(t_1) - T(t_2)\| \|(\phi(s) - h(0,\phi))\| + \left\|h\left(t_1, y_{t_1} + \hat{\phi}_{t_1}\right) - h\left(t_2, y_{t_2} + \hat{\phi}_{t_2}\right)\right\|$$

$$+\int_0^{t_1}\|A[T(t_1-s)-T(t_2-s)]\|\left(L(|t-s|)+\left\|y_{s_1}+\phi_{s_1}-y_{s_2}+\phi_{s_2}\right\|\right)ds$$
$$+\int_{t_1}^{t_2}\|AT(t_2-s)\|\left(L(|t-s|)+\left\|y_{s_1}+\hat{\phi}_{s_1}-y_{s_2}+\hat{\phi}_{s_2}\right\|\right)ds$$
$$+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|\left[M_3\|u(s)\|+M_2\left(|t-s|+\left\|y_s+\hat{\phi}_s\right\|\right)+N_1+N(s)\right]ds$$
$$+\int_{t_2}^{t_1}\|T(t_2-s)\|\left[M_3\|u(s)\|+M_2\left(|t-s|+\left\|y_s+\hat{\phi}_s\right\|\right)+N_1+N(s)\right]ds \qquad (3.3)$$

The right hand side of (3.3) is independent of $y\in B_k$ and tends to zero as $t_1-t_2\to 0,$ since h is completely continuous and the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology. Thus Γ maps B_k into an equicontinuous family of functions.

We have already shown that ΓB_k is equicontinuous and uniformly bounded collection. Next we show that ΓB_k is compact. Since we have shown that ΓB_k is an equicontinuous collection, it is sufficient by Arzela-Ascoli theorem to show that Γ maps B_k into precompact set in X. Let $o<t\le b$ be fixed and let ε be a real number satisfying $0<\varepsilon<t$. For $y\in B_k$, we define

$$(\Gamma_\varepsilon y)(t)=T(t)[\phi(0)-h(0,\phi)]+h\left(t-\varepsilon,y_{t-\varepsilon}+\hat{\phi}_{t-\varepsilon}\right)+\int_0^{t-\varepsilon}AT(t-s)h\left(s,y_s+\hat{\phi}_s\right)ds$$
$$+\int_0^{t-\varepsilon}T(t-s)\left[Bu(s)+f\left(s,y_s+\hat{\phi}_s\right)+q(s)+\int_0^s g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau\right]ds$$
$$=T(t)[\phi(0)-h(0,\phi)]+h\left(t-\varepsilon,y_{t-\varepsilon}+\hat{\phi}_{t-\varepsilon}\right)+T(\varepsilon)\int_0^{t-\varepsilon}AT(t-s-\varepsilon)h\left(s,y_s+\hat{\phi}_s\right)ds$$
$$+T(\varepsilon)\int_0^{t-\varepsilon}T(t-s-\varepsilon)\left[Bu(s)+f\left(s,y_s+\hat{\phi}_s\right)+q(s)+\int_0^s g\left(s,\tau,y_\tau+\hat{\phi}_s\right)d\tau\right]ds$$

Since $T(t)$ is a compact operator, the set $Y_\varepsilon(t)=\{(\Gamma_\varepsilon y)(t): y\in B_k\}$ is precompact in X for every ε, $0<\varepsilon<t$ Moreover, for every $y\in B_k$ we have

$$\|(\Gamma y)(t)-(\Gamma_\varepsilon y)(t)\|\le\left\|h\left(t,y_t+\hat{\phi}_t\right)-h\left(t-\varepsilon,y_{t-\varepsilon}+\hat{\phi}_{t-\varepsilon}\right)\right\|+\int_{t-\varepsilon}^{t}\left\|AT(t-s)h\left(s,y_s+\hat{\phi}_s\right)\right\|ds$$
$$+\int_{t-\varepsilon}^{t}T(t-s)\left[Bu(s)+f\left(s,y_s+\hat{\phi}_s\right)+q(s)+\int_0^s g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau\right]ds$$
$$\le\left\|h\left(t,y_t+\hat{\phi}_t\right)-h\left(t-\varepsilon,y_{t-\varepsilon}+\hat{\phi}_{t-\varepsilon}\right)\right\|+L\int_{t-\varepsilon}^{t}\left\|AT(t-s)\left(y_s+\hat{\phi}_s\right)\right\|$$
$$+\int_{t-\varepsilon}^{t}T(t-s)\left[M_3\|u(s)\|+M_2\left(|t-s|+\left\|y_s+\hat{\phi}_s\right\|\right)+N_1+N(s)\right]ds$$

Therefore, there are precompact sets arbitrary close to the set $\{(\Gamma y)(t): y\in B_k\}$. Hence the set $\{(\Gamma y)(t): y\in B_k\}$ is precompact in X.

It remains to show that $\Gamma:\Re_0\to\Re_0$ is continuous. Let $\{y_n\}_{n\ge1}\subset\Re_0$ with $y_n\to y$ in $\Re_0$. We have

$$\|(\Gamma y_n)(t)-(\Gamma y)(t)\|\le\left\|h\left(t,y_{n_t}+\hat{\phi}_t\right)-h\left(t,y_t+\hat{\phi}_t\right)\right\|+\int_0^t\|AT(t-s)\|\left\|h\left(s,y_{n_s}+\hat{\phi}_s\right)-h\left(s,y_s+\hat{\phi}_s\right)\right\|ds$$

$$+\int_0^t\|T(t-s)\|\left[\left\|f\left(s,y_{n_s}+\hat{\phi}_s\right)-f\left(s,y_s+\hat{\phi}_s\right)\right\|+\int_0^s\left\|g\left(s,\tau,y_{n_\tau}+\hat{\phi}_\tau\right)-g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)\right\|d\tau\right]ds\to 0$$

as $n\to\infty$.

Thus Γ is continuous. This completes the proof that Γ is a completely continuous operator. Moreover, the set

$$\xi(\Gamma)=\{y\in\Re_0 : y=\lambda\Gamma y.\ \ 0<\lambda<1\}$$

is bounded in $\Re$, since for every y in $\xi(\Gamma)$, the function $x(t) = y(t) + \hat{\phi}(t), t \in (-\infty, a]$ is a solution of equation (2.1) for which we have proved to be bounded. Consequently by Schaefer's theorem, the operator Γ has a fixed point in $\bar{y}_0$ in $\Re_0$. Then $\bar{x} = \bar{y} + \hat{\phi}$ is a mild solution of equation (2.1).

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