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ON HYPERBOLICALLY KAEHLERIAN BI-RECURRENT AND BI-SYMMETRIC SPACES

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***Abstract**: Tachibana (1967) have studied on the Bochner curvature tensor. Singh (1971-72) studied on Kaehlerian recurrent and Ricci-recurrent spaces of second order. Further, Negi and Rawat (1994) have been studied some bi-recurrent and bi-sym metric properties in a Kaehlerian space.*

In the present paper, we have been studied Hyperbolically Kaehlerian bi-recurrent and bi-symmetric spaces also several theorems have been established and proved therein.
***Keywords:** Bi-recurrent, bi-symmetric, Hyperbolically Kaehlerian Space, Kaehlerian space, Sasakian space.*

I. Introduction

Spaces with additional structures which arise in theoretical Physics play an important part in the theory of Riemannian spaces V_n . Such spaces are, in particular, "**Classical"** Kaehlerian and Sasakian spaces as well as hyperbolically Kaehlerian and Hyperbolically Sasakian spaces.

Definition (1.1): A four- dimensional Riemannian space K_n is called a hyperbolically Kaehlerian space if, along with Riemannian metric tensor g_{ij} , a complex structure tensor F_i^h satisfies the following conditions:

$$F_k^h F_i^k = \delta_i^h , \qquad \dots (1.1)$$

$$g_{ki} F_j^k + g_{kj} F_i^k = 0 , \qquad \dots (1.2)$$

$$F_{i,j}^h = 0 , \qquad \dots (1.3)$$

where the comma (,) followed by an index denotes the operator of covariant differentiation w.r.to the Riemannian metric tensor.

Definition (1.2): An odd-dimensional Riemannian space S_n is called a hyperbolically Sasakian space if, along with metric tensor g_{ij} , a complex structure tensor F_i^h satisfies the following conditions :

$$F_k^h F_i^k = \delta_i^h - X^h X_i , \qquad \dots (1.4)$$

$$F_k^h X^k = 0 , \qquad \dots (1.5)$$

$$X^k X_k = 1 , \qquad \dots (1.6)$$

$$g_{ki} F_j^k + g_{kj} F_i^k = 0 , \qquad \dots (1.7)$$

$$F_{i,j}^h = X^h g_{ij} - \delta_j^h X_i , \qquad \dots (1.8)$$

where $X_i \equiv X^k g_{ki}$ is some vector.

Differentiating (1.4), it is easy to establish that $F_i^h = X_{,i}^k$. This definition of Sasakian spaces is over determined.

The Riemannian curvature tensor field R_{ijk}^h is defined as

$$R_{ijk}^h = \partial_i \left\{ {h \atop j\ k} \right\} - \partial_j \left\{ {h \atop i\ k} \right\} + \left\{ {h \atop i\ a} \right\} \left\{ {a \atop j\ k} \right\} - \left\{ {h \atop j\ a} \right\} \left\{ {a \atop i\ k} \right\} ,$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes the real local coordinates.

The Ricci tensor and the Scalar curvature are respectively given by

$$R_{ij} = R_{aij}^a \text{ and } R = g^{ij} R_{ij} .$$

If we define a tensor S_{ij} by

$$S_{ij} = F_i^a R_{aj} , \qquad \dots (1.9)$$

Then we have

$S_{ij} = -S_{ji}$, ... (1.10)

$F_i^a S_{aj} = -S_{ia} F_j^a$, ... (1.11)

and

$F_i^a S_{jk,a} = R_{ji,k} - R_{ki,j}$ . ...(1.12)

It has been verified in Yano([5]) pages 63, 68 that the metric tensor g_{ij} and the Ricci-tensor denoted by R_{ij} are hybrid in i and j. Therefore, we get

$g_{ij} = g_{rs} F_i^r F_j^s$, ... (1.13)

and

$R_{ij} = R_{rs} F_i^r F_j^s$, ... (1.14)

The Holomorphically Projective curvature tensor P_{ijk}^h is given by

$P_{ijk}^h = R_{ijk}^h + \frac{1}{(n+2)} \left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + 2S_{ij}F_k^h\right)$, ... (1.15)

The Tachibana H-Concircular curvature tensor and the Weyl-Conformal curvature tensors are respectively given by

$T_{ijk}^h = R_{ijk}^h + \frac{R}{n(n+2)} \left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right)$, ... (1.16)

and

$C_{ijk}^h = \frac{1}{(n-)} \left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + g_{ik}R_j^h - g_{jk}R_i^h\right) - \frac{R}{(n-1)(n-2)} \left(g_{ik}\delta_j^h - g_{jk}\delta_i^h\right)$, ... (1.17)

There is a Weyl-Concircular curvature tensor given by (Sinha, 1971)

$Z_{ijk}^h = R_{ijk}^i + \frac{R}{n(n-1)} \left(g_{ik}\delta_j^h - g_{jk}\delta_i^h\right)$, ... (1.18)

If, we put

$L_{ij} = R_{ij} - \frac{R}{n} g_{ij}$ ... (1.19)

and

$M_{ij} = F_i^a L_{aj} = S_{ij} - \frac{R}{n} F_i$ ... (1.20)

Then from (1.15), (1.16), (1.16), (1.19) and (1.20), we get

$P_{ijk}^h = T_{ijk}^h + \frac{1}{(n+2)} \left(L_{ik}\delta_j^h - L_{jk}\delta_i^h + M_{ik}F_j^h - M_{jk}F_i^h + 2M_{ij}F_k^h\right)$, ... (1.21)

and with the help of (1.17), (1.18), (1.19) and (1.20), we have

$C_{ijk}^h = Z_{ijk}^h + \frac{1}{(n-2)} \left(L_{ik}\delta_j^h - L_{jk}\delta_i^h + g_{ik}L_j^h - g_{jk}L_i^h\right)$. ... (1.22)

Now, we shall use the following:

Definition (1.3). A hyperbolically Kaehlerian space K_n is said to be bi-recurrent, if we have

$R_{ijk,ab}^h - \lambda_{ab}R_{ijk}^h = 0,$ or, equivalently $R_{ijkl,ab} - \lambda_{ab}R_{ijkl} = 0.$ ... (1.23)

for some non-zero tensor field λ_{ab} , and is known as recurrence tensor field.

A hyperbolically Kaehlerian space whose Ricci-tensor R_{ij} satisfies the equation

$R_{ij,ab} - \lambda_{ab}R_{ij} = 0,$ ... (1.24)

for some non-zero tensor λ_{ab} , is called hyperbolically Kaehlerian Ricci-bi-recurrent space. Multiplying the above equation by g^{ij} , we have

$R_{,ab} - \lambda_{ab}R = 0.$ ... (1.25)

II. Hyperbolically Kaehlerian Spaces With Bi-Recurrent Properties

Definition (2.1). A hyperbolically Kaehlerian space satisfying the relation

$P_{ijk,ab}^h - \lambda_{ab}P_{ijk}^h = 0,$ or, equivalently $P_{ijkl,ab} - \lambda_{ab}P_{ijkl} = 0,$ ... (2.1)

For some non-zero tensor field λ_{ab} , will be called hyperbolically Kaehlerian projective bi-recurrent space.

Definition (2.2). A hyperbolically Kaehlerian space satisfying the relation

$T_{ijk,ab}^h - \lambda_{ab}T_{ijk}^h = 0,$ or, equivalently $T_{ijkl,ab} - \lambda_{ab}T_{ijkl} = 0,$ ... (2.2)

For some non-zero tensor field λ_{ab} , will be called hyperbolically Kaehlerian space with Tachibana H-Concircular bi-recurrent space.

Definition (2.3). A hyperbolically Kaehlerian space satisfying the relation

$C_{ijk,ab}^h - \lambda_{ab}C_{ijk}^h = 0,$ or, equivalently $C_{ijkl,ab} - \lambda_{ab}C_{ijkl} = 0,$ ... (2.3)

For some non-zero tensor field λ_{ab} , will be called hyperbolically Kaehlerian space with bi-recurrent Weyl-Conformal curvature tensor.

Definition (2.4). A hyperbolically Kaehlerian space satisfying the relation

$$Z^h_{ijk,ab} - \lambda_{ab} Z^h_{ijk} = 0, \text{ or, equivalently } Z_{ijkl,ab} - \lambda_{ab} Z_{ijkl} = 0, \qquad \dots (2.4)$$

For some non-zero recurrence tensor field λ_{ab}, will be called hyperbolically Kaehlerian space with bi-recurrent Weyl-Concircular curvature tensor.

Now, we have the following:

Theorem (2.1) : If a hyperbolically Kaehlerian space satisfying any two of the following properties:

(i) the space is hyperbolically Kaehlerian Ricci-bi-recurrent ,
(ii) the space is hyperbolically Kaehlerian projective bi-recurrent ,
(iii) the space is hyperbolically Kaehlerian Tachibana H-Concircular bi-recurrent ,then it must also satisfy the third.

Proof. Differentiating (1.21) covariantly w.r.to x^a, again differentiate the result thus obtained covariantly w.r.to x^b, we have

$$P^h_{ijk,ab} = T^h_{ijk,ab} = \frac{1}{(n+2)}\left(L_{ik,ab}\delta^h_j - L_{jk,ab}\delta^h_i + M_{ik,ab}F^h_j - M_{jk,ab}F^h_i + 2M_{ij,ab}F^h_k\right), \qquad \dots (2.5)$$

Multiplying (1.21) with λ_{ab} and subtracting the result thus obtained from (2.5), we have

$$P^h_{ijk,ab} - \lambda_{ab}P^h_{ijk} = T^h_{ijk,ab} - \lambda_{ab}T^h_{ijk} + \frac{1}{(n+2)}\left[\left(L_{ik,ab} - \lambda_{ab}L_{ik}\right)\delta^h_j - \left(L_{jk,ab} - \lambda_{ab}L_{jk}\right)\delta^h_i + \left(M_{ik,ab} - \lambda_{ab}M_{ik}\right)F^h_j - \left(M_{jk,ab} - \lambda_{ab}M_{jk}\right)F^h_i + 2\left(M_{ij,ab} - \lambda_{ab}M_{ij}\right)F^h_k\right]$$

…(2.6)

The statement of the above theorem follows in view of equations (1.24), (1.25), (2.1), (2.2), (1.19), (1.20) and (2.6).

Theorem (2.2). If a hyperbolically Kaehlerian space satisfies any two of the following properties:

(i) the space is hyperbolically Kaehlerian Ricci-bi-recurrent ,
(ii) the space is hyperbolically Kaehlerian space with bi-recurrent Weyl-Conformal curvature tensor,
(iii) the space is hyperbolically Kaehlerian space with bi-recurrent Weyl-Concircular curvature tensor , then it must also satisfy the third.

Proof. A Hyperbolically Kaehlerian Ricci –bi-recurrent space, a Hyperbolically Kaehlerian space with bi-recurrent Weyl-Conformal curvature tensor and hyperbolically Kaehlerian space with bi-recurrent Weyl-Concircular curvature tensor are respectively characterized by the equations (1.24), (2.3) and (2.4).

Differentiating (1.22) covariantly w.r.to x^a, again differentiate the result thus obtained covariantly w.r.to x^b, we have

$$C^h_{ijk,ab} = Z^h_{ijk,ab} + \frac{1}{(n-2)}\left(L_{ik,ab}\delta^h_j - L_{jk,ab}\delta^h_i + g_{ik}L^h_{j,ab} - g_{jk}L^h_{i,ab}\right), \qquad \dots (2.7)$$

Multiplying (1.22) with λ_{ab} and subtracting the result thus obtained from (2.7), we have

$$C^h_{ijk,ab} - \lambda_{ab}C^h_{ijk} = Z^h_{ijk,ab} - \lambda_{ab}Z^h_{ijk} + \frac{1}{(n-2)}\left[\left(L_{ik,ab} - \lambda_{ab}L_{ik}\right)\delta^h_j - \left(L_{jk,ab} - \lambda_{ab}L_{jk}\right)\delta^h_i + \left(L^h_{j,ab} - \lambda_{ab}L^h_j\right)g_{ik} - \left(L^h_{i,ab} - \lambda_{ab}L^h_i\right)g_{jk}\right], \qquad \dots (2.8)$$

The statement of the above theorem follows in view of (1.19), (1.20) (1.24), (2.3), (2.4) and (2.8).

Theorem (2.3). Every hyperbolically Kaehlerian bi-recurrent space is a hyperbolically Kaehlerian space with Tachibana H-Concircular bi-recurrent space.

Proof. Differentiating (1.16) covariantly w.r.to x^a, again differentiate the result thus obtained covariantly w.r.t. x^b, we have

$$T^h_{ijk,ab} = R^h_{ijk,ab} + \frac{R_{,ab}}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \qquad \dots (2.9)$$

Multiplying (1.16) by λ_{ab} and subtracting the result thus obtained from (2.9), we have

$$T^h_{ijk,ab} - \lambda_{ab}T^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab}R^h_{ijk} + \frac{(R_{,ab} - \lambda_{ab})}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right),$$

…
(2.10)

Now, let the space be hyperbolically Kaehlerian bi-recurrent, then equations (1.23), (1.24) and (1.25) are satisfied.

Making use of equations (1.23) and (1.25) in (2.10), we have

$$T^h_{ijk,ab} - \lambda_{ab}T^h_{ijk} = 0,$$

which shows that the space is hyperbolically Kaehlerian space with Tachibana H-Concircular bi-recurrent space.

III. Hyperbolically Kaehlerian Spaces With Bi-Symmetric Properties

Definition (3.1). A hyperbolically Kaehlerian space is said to be bi-symmetric if it satisfies the relation

$$R^h_{ijk,ab} = 0, \text{ or , equivalently } R_{ijkl,ab} = 0, \quad \dots (3.1)$$

Obviously, a hyperbolically Kaehlerian bi-symmetric space is said to be hyperbolically Kaehlerian Ricci-bi-symmetric space if

$$R_{ij,ab} = 0, \quad \dots (3.2)$$

Multiplying the above equation by g^{ij}, we get

$$R_{,ab} = 0. \quad \dots (3.3).$$

Definition (3.2). A hyperbolically Kaehlerian space satisfying the relation

$$P^h_{ijk,ab} = 0, \text{ or, equivalently } P_{ijkl,ab} = 0, \quad \dots (3.4)$$

is called a hyperbolically Kaehlerian projective bi-symmetric space.

Definition (3.3). A hyperbolically Kaehlerian space satisfying the relation

$$T^h_{ijk,ab} = 0, \text{ or, equivalently } T_{ijkl,ab} = 0, \quad \dots (3.5)$$

will be called hyperbolically Kaehlerian space with Tachibana H-Concircular bi-symmetric space.

Definition (3.4). A hyperbolically Kaehlerian space satisfying the relation

$$C^h_{ijk,ab} = 0, \text{ or, equivalently } C_{ijkl,ab} = 0, \quad \dots (3.6)$$

will be called hyperbolically Kaehlerian space with bi-symmetric Weyl-Conformal curvature tensor.

Definition (3.5). A hyperbolically Kaehlerian space satisfying the relation

$$Z^h_{ijk,ab} = 0, \text{ or, equivalently } Z_{ijkl,ab} = 0, \quad \dots (3.7)$$

is called hyperbolically Kaehlerian space with bi-symmetric Weyl-Concircular curvature tensor.
Now, we have the following :

Theorem (3.1). If a hyperbolically Kaehlerian space satisfies any two of the following properties:
(i) the space is hyperbolically Kaehlerian Ricci-bi-symmetric,
(ii) the space is hyperbolically Kaehlerian projective bi-symmetric ,
(iii) the space is hyperbolically Kaehlerian Tachibana H-Concircular bi-symmetric, then it must also satisfy the third.

Proof. A hyperbolically Kaehlerian Ricci-bi-symmetric space, a hyperbolically Kaehlerian Projective bi-symmetric space and hyperbolically Kaehlerian space with Tachibana H-Concircular bi-symmetric space are respectively characterized by (3.2), (3.4) and (3.5).

The statement of the above theorem follows in view of (2.5), (3.2), (3.4) and (3.5).

Theorem (3.2) . If a hyperbolically Kaehlerian space satisfies any two of the following properties:
(i) the space is hyperbolically Kaehlerian Ricci-bi-symmetric,
(ii) the space is hyperbolically Kaehlerian space with bi-symmetric Weyl-Conformal curvature tensor,
(iii) the space is hyperbolically Kaehlerian space with bi-symmetric Weyl-Concircular curvature tensor , then it must also satisfy the third.

Proof. A Hyperbolically Kaehlerian Ricci –bi-symmetric space, a Hyperbolically Kaehlerian space with bi-symmetric Weyl - Conformal curvature tensor and hyperbolically Kaehlerian space with bi-symmetric Weyl-Concircular curvature tensor are respectively characterized by (3.2), (3.6) and (3.7).

The statement of the above theorem follows in view of (2.7), (3.2), (3.6) and (3.7).

Theorem (3.3). Every hyperbolically Kaehlerian bi-symmetric space is a hyperbolically Kaehlerian space with Tachibana H-Concircular bi-symmetric space.

Proof. From (2.9), it follows that in a hyperbolically Kaehlerian bi-symmetric space, the Tachibana H-Concircular curvature tensor satisfies

$$T^h_{ijk,ab} = 0,$$

which shows that the space is hyperbolically Kaehlerian space with Tachibana H-Concircular bi-symmetric space.

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