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An Error Estimation of the Tau Method for Some Class of Ordinary Differential Equations.

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Abstract: *This paper is concerned with error estimation of the integrated variant of the tau method for Initial Value Problems (IVPs) for the class of equations for which* $m + s \leq 3$ *where m and s are, respectively, the order and the number of over determination. Some general results obtained are applied to some problems. The numerical evidences show that the order of the tau approximant is closely captured.*

Key Words: *Tau methods, error estimation, overdetermination, variant, order, Initial. Value Problem (IVP)*

I. Introduction

The tau method proposed by Lanczos (1935) seeks an approximant

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \quad n < +\infty \tag{1.1}$$

of the solution y(x) to the linear m-th order of ODE:

$$Ly(x) \equiv \sum_{r=0}^{m}\left(\sum_{k=0}^{N_r} p_{rk}\, x^k\right) y^{(r)}(x) = f(x), \quad a \leq x \leq b \qquad (1.2a)$$

$$L^* y(a) \equiv y^{(a)}(a) = \propto_r, r = 1(1)m - 1 \qquad (1.2b)$$

which satisfies the corresponding perturbed problem;

$$Ly_n(x) = f(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r}\, T_{n-m+r+1}(x) \qquad (1.3a)$$

$$L^* y_n(x_{rk}) = \propto_k, \qquad k = 1(1)m - 1 \qquad (1.3b)$$

Where $a_{rk}, x_{rk}, \propto_k$ $r = 0(1)m - 1, k = 1(1)m$ are given real numbers, f(x) is a polynomial function or sufficiently close approximants of given real function;

$$T_k(x) = \cos\left(x\cos^{-1}\left(\frac{2x-a-b}{b-a}\right)\right) \equiv \sum_{r=0}^{k} C_r^{(k)} x^r \qquad (1.4)$$

is the k-th shifted chebyshev polynomial valid in [a,b] and τ 's are fixed tau parameters to be determined along with the coefficients of $y_n(x)$;

$$S = max\{N_r - r : N_r, 0 \leq r \leq m\} \geq 0 \qquad (1.5)$$

is the number of over determination of (1.2).

An attempt to improve the accuracy of $y_n(x)$ gave rise to the integrated variant of (1.3) in the form

$$\begin{aligned} I_L(y_n(x)) &\equiv \iint \ldots^m \ldots \int Ly_n(x)\,dxdx \ldots.dx \\ &= \iint \ldots^m \ldots \int f(x)\,dxdx \ldots dx \\ &+ \sum_{r=0}^{m+s+1} \tau_{m+s-r} T_{n-m+r+2} + C_{m-1}(x) \end{aligned} \qquad (1.6)$$

where $\iint \ldots^m \ldots \int Ly_n(x)dxdx \ldots.dx$ denotes an m-times integration and $C_{m-1}(x)$ an arbitrary polynomial of degree (m-1) arising from constants of integration. The higher order perturbation term in (1.6) accurate for the improved accuracy of the corresponding approximation $y_n(x)$.

II. Error Estimation of the Tau method

Error estimation of the tau method has been reported in literature(see Lanczos, Fox, Onumanyi and Ortiz). Adeniyi et al also reported an error estimation of the tau method (1.3) which was based on a modification of the error of the Lanczos economization
process. By this the error polynomial.

$$(e_n(x))_{n+1} = \frac{(x-a)^m \varphi_n T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \cong e_n(x) = y(x) - y_n(x) \quad (2.1)$$

Satisfies the perturbed error DE

$$L(e_n(x))_{n+1} = -\sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+2}(x) \quad (2.2)$$

Corresponding to the error DE

$$L(e_n(x))_{n+1} = -\sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \quad (2.3)$$

and where φ_n is a parameter to be determined along with the $\hat{\tau}$'s in (2.2) an error estimation of the integrated variant, we have from (2.3) that

$$I_L(e_n(x))_{n+1} = -\iint \dots^m \dots \int \left(\sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \right) dx \dots dx + \sum_{r=0}^{m+s-1} \tau_{m+s-r+3}(x) \quad (2.4)$$

We insert (2.2) into (2.4) and then equate coefficients of powers of $x^{n+m+s+1}, x^{n+m+s}, \dots., x^{n-m}$, determination of φ_n. A forward equation process is recommended for this purpose. Once φ_n is determined then

$$\varepsilon^* = \max_{a \le x \le b} |(e_n(x))_{n+1}| = \frac{|\varphi_n|}{\left|C_{n-m+1}^{(n-m+1)}\right|} \cong max|e_n(x)| = \varepsilon \quad (2.5)$$

III. Derivation of Error Estimate for the Integrated Formulation.

We consider here the error estimation for the integration formulation of the tau method for class of problem (1.2) where m + s≤3.

3.1 The Case m=1,s=0

For this case we have from (1.2) that

$$Ly(x) = \left(P_{10} + P_{11}(x)\right)y'(x) + P_{00}y(x) = f(x)$$

$$\equiv \sum_{r=0}^{F} f_r x^r \;, \qquad \mathrm{y(a)} = \alpha_0 \quad (3.1)$$

with the corresponding perturbed integrated form of (3.1)

$$I_L(e_n(x))_{n+1} \equiv \int_0^x \left[\left(P_{10} + P_{11}(x)\right)\left(e_n'(x)\right)_{n+1} + P_{00}\left(e_n(x)\right)_{n+1}\right] dx = -\tau_1 \int_0^x \left(\sum_{r=0}^{n} C_r^{(n)} x^r\right) dx + \hat{\tau}_1 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r \quad (3.2)$$

where,

$$(e_n(x))_{n+1} = \frac{\varphi_n x T_n(x)}{C_n^{(n)}} = \frac{\varphi_n}{C_n^{(n)}} \sum_{r=0}^{n} C_r^{(n)} x^{r+1} \quad (3.3)$$

This leads to

$$\frac{\varphi_n}{k_1}[\lambda_1 x^{n+2} + \lambda_2 x^{n+1} + \cdots] = \hat{\tau}_1 C_{n+2}^{(n+2)} x^{n+2} + \left[\hat{\tau}_1 C_{n+1}^{(n+2)} - \frac{\tau_1 C_n^{(n)}}{n+1}\right] x^{n+1} \quad (3.4)$$

where,

$$\lambda_1 = \left\{\frac{P_{00} + (n+1)P_{11}}{m+2}\right\} k_1 \quad (3.5)$$

$$\lambda_2 = P_{10}k_1 + \left[\frac{P_{00} + xP_{11}}{n+2}\right] k_2 \quad (3.6)$$

By equation corresponding coefficients of x^{n+2} and x^{n+1} in (3.2) we have

$$\hat{\tau}_1 C_{n+2}^{(n+2)} = \frac{\lambda_1 \varphi_n}{k_1} \quad (3.7)$$

$$\tilde{\tau}_1 C_{n+2}^{(n+2)} - \frac{k_1 \tau_1}{n+2} = \frac{\lambda_2 \varphi_n}{k_1} \quad (3.8)$$

From (3.7) we get

$$\tilde{\tau}_1 = \frac{\lambda \varphi_n}{k_1 C_{n+2}^{(n+2)}} \quad (3.9)$$

Hence , (3.8) gives

$$\varphi_n = \frac{R_1^2 \tau_1}{(n+1)R_2} \quad (3.10)$$

where,

$$R_2 = \lambda_2 - \frac{\lambda_1 C_{n+1}^{(n+2)}}{C_{n+2}^{(n+2)}} \quad (3.11)$$

By letting $R_1 = \lambda_1$, R_2 can be expressed in the recursive form.

$$R_2 = \lambda_2 - \frac{C_{n+1}^{(n+2)} R_1}{C_{n+2}^{(n+2)}} \quad (3.12)$$

$R_1 = \lambda_1$,

$$\lambda_1 = \left\{\frac{P_{00} + (n+1)P_{11}}{m+2}\right\} k_1 \quad (3.13a)$$

$$\lambda_2 = P_{10}k_1 + \left[\frac{P_{00} + xP_{11}}{n+2}\right] k_2 \quad (3.13b)$$

$k_1 = C_n^{(n)}$, $\quad k_2 = C_{n-1}^{(n)}$

3.2 The case m=1, s=1

The problem to be considered in this case is

$$y(x) \equiv (P_{10} + P_{11}x + P_{12}x^2)y'(x) + (P_{00} + P_{01}x)y'(x) = \sum_{r=0}^{F} f_r x^r, \quad a \le x \le b \quad (3.14a)$$

$$y(a) = \alpha_0 \quad (3.14b)$$

The associated perturbed error equation to be considered is

$$I_L(e_n(x))_{n+1} \equiv \int_0^{\alpha} \left[(P_{10} + P_{11}(x))\left(e_n'(x)\right)_{n+1} + (P_{00} + P_{01}x)_{n+1}\right] dx$$
$$= -\int_{r=0}^{n+1} \left[\tau_1 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r + \tau_1 \sum_{r=0}^{n} C_r^{(n)} x^r\right] dx + \hat{\tau}_1 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r$$
$$+ \hat{\tau}_2 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r \quad (3.15)$$

are $(e_n(x))_{n+1}$is again defined by (3.3) We solve (3.15) by forward elimination process to obtain

$$\varphi_n = -\frac{R_1^2\tau_1}{(n+1)R_3} \qquad (3.16)$$

whereR_3 is given recursively by

$$R_3 = \lambda_3 - \frac{C_{n+1}^{(n+3)}}{C_{n+3}^{(n+3)}}R_1 - \frac{C_{n+1}^{(n+2)}}{C_{n+2}^{(n+2)}}R_2 \qquad (3.17a)$$

$$R_2 = \lambda_2 - \frac{C_{n+2}^{(n+3)}}{C_{n+3}^{(n+3)}}R_1 \qquad (3.17b)$$

$$R_1 = \lambda_1 \qquad (3.17c)$$

where,

$$\lambda_1 = \left[\frac{P_{01} + (n+1)P_{12}}{n+3}\right]R_1$$

$$\lambda_2 = \left[\frac{P_{00} + (n+1)P_{11}}{n+2}\right]R_1 + \left[\frac{P_{01} + nP_{12}}{n+2}\right]R_2$$

$$\lambda_3 = P_{10}R_1 + \left[\frac{P_{00} + nP_{11}}{n+1}\right]R_2 + \left[\frac{P_{01} + (n-1)P_{12}}{n+1}\right]R_3 \qquad (3.18)$$

$R_1 = C_n^{(n)}, R_2 = C_{n-1}^{(n)}, R_3 = C_{n-2}^{(n)}$

3.3 The case m=1, s=2;

In this case, the problem to the considered is

$$Ly(x) \equiv (P_{10} + P_{11}x + P_{12}x^2 + P_{13}x^3)y'(x) + (P_{00} + P_{01}x + P_{02}x^2)y(x) = \sum_{r=0}^{F} f_r x^r \ , \ a \le x \le b \qquad (3.19)$$

where the perturbed associated error equation is

$$I_L(e_n(x))_{n+1} \equiv \int_0^x \left[(P_{10} + P_{11}u + P_{12}u^2 + P_{13}u^3)\left(e_n'(x)\right)_{n+1} + (P_{00} + P_{01}x + P_{02}x^2)(e_n(u))_{n+1}\right]dx$$
$$= -\int_0^x \left[\tau_1 \sum_{r=0}^{n+2} C_r^{(n+2)}u^r + \tau_2 \sum_{r=0}^{n+1} C_r^{(n+1)}u^r + \tau_3 \sum_{r=0}^{n} C_r^{(n)}u^r\right]du + \hat{\tau}_1 \sum_{r=0}^{n+4} C_r^{(n+4)}x^r$$
$$+ \hat{\tau}_2 \sum_{r=0}^{n+3} C_r^{(n+3)}x^r + \hat{\tau}_3 \sum_{r=0}^{n+2} C_r^{(n+2)}x^r \qquad (3.20)$$

Where $(e_n(x))_{n+1}$ is as defined in (3.3) . We then solve the (3.20) by equating the corresponding coefficient of $x^{n+4}, x^{n+3}, x^{n+2} and x^{n+1}$ in (3.20) to have

$$\varphi_n = \left[\frac{C_{n+1}^{(n+3)}C_{n+2}^{(n+2)}}{(n+3)C_{n+3}^{n+3}} - \frac{C_n^{(n+2)}}{(n+1)}\right]\frac{\tau_1 k_1}{R_4} - \frac{k_1^2\tau_3}{(n+1)R_4} \qquad (3.21)$$

whereR_4 is obtained recursively by

$$R_4 = \lambda_4 - \frac{C_{n+1}^{n+4}R_1}{C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{n+3}R_2}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{n+2}R_3}{C_{n+2}^{(n+2)}}$$

$$R_3 = \lambda_3 - \frac{C_{n+1}^{n+4}R_1}{C_{n+4}^{(n+4)}} - \frac{C_{n+2}^{n+3}R_2}{C_{n+3}^{(n+3)}}$$

$$R_3 = \lambda_2 - \frac{C_{n+1}^{n+4}R_1}{C_{n+4}^{(n+4)}}$$

$R_1 = \lambda_1$ (3.22)

where, $\lambda_1 = \left[\frac{P_{02} + (n+1)P_{13}}{n+3}\right] k_1$

$$\lambda_2 = \left[\frac{P_{01} + (n+1)P_{12}}{n+3}\right] k_1 + \left[\frac{P_{02} + nP_{13}}{n+3}\right] k_2$$

$$\lambda_3 = \left[\frac{P_{00} + (n+1)P_{11}}{n+2}\right] k_1 + \left[\frac{P_{01} + nP_{12}}{n+2}\right] k_2 + \left[\frac{P_{02} + (n-1)P_{13}}{n+2}\right] k_3$$

$$\lambda_4 = P_{10} k_1 + \left[\frac{P_{00} + nP_{11}}{n+1}\right] k_2 + \left[\frac{P_{01} + (n-1)P_{12}}{n+1}\right] k_3 + \left[\frac{P_{02} + (n-2)P_{13}}{n+1}\right] k_4 \quad (3.23)$$

$k_1 = C_n^{(n)}, k_2 = C_{n-1}^{(n)}, k_3 = C_{n-2}^{(n)}, k_4 = C_{n-3}^{(n)}$

3.4 The case m=2, s=0

From (1.2), we have for m=2 and s=0 the problem

$$Ly(x) \equiv (P_{20} + P_{21}x + P_{22}x^2)y''(x) + (P_{10} + P_{11}x)y'(x) + P_{00}y(x) = \sum_{r=0}^{F} f_r x^r \ , \quad a \le x \le b \quad (3.24)$$

$$y(a) = \alpha_0 \, , y'(a) = \alpha_1$$

With the associated perturbed error problem

$$I_L\big(e_n(x)\big)_{n+1} \equiv \int_0^x \int_0^u \Big[(P_{20} + P_{21}t + P_{22}t^2)(e_n''(t))_{n+1} + (P_{10} + P_{11}t)(e_n'(t))_{n+1} + P_{00}\big(e_n(t)\big)_{n+1}\Big] dt du = -\int_0^x \int_0^u \Big[\tau_1 \sum_{r=0}^{n} C_r^{(n)} t^r + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)} t^r\Big] dt du + \hat{\tau}_1 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r + \hat{\tau}_2 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r \quad (3.25)$$

Where,
$(e_n(x))_{n+1}$ in this case from (1.7) is given by

$$\big(e_n(x)\big)_{n+1} = \frac{\varphi x^2 T_{n-1}(x)}{C_{n-1}^{(n-1)}} = \frac{\varphi_n}{C_{n-1}^{(n-1)}} \sum_{r=0}^{n-1} C_r^{(n-1)} x^{r+2} \quad (3.26)$$

This gives,

$$\frac{\varphi_n}{\kappa_1}[\lambda_1 x^{n+3} + \lambda_2 x^{n+2} + \lambda_3 x^{n+1} + \cdots] = \hat{\tau}_1 C_{n+3}^{(n+3)} x^{n+3} \left[\hat{\tau}_1 C_{n+2}^{(n+3)} + \hat{\tau}_2 C_{n+2}^{(n+2)} - \frac{\hat{\tau}_1 C_n^{(n)}}{(n+1)(n+2)}\right] x^{n+2} + \left[\hat{\tau}_1 C_{n+1}^{(n+3)} + \hat{\tau}_2 C_{n+1}^{(n+2)} - \frac{\hat{\tau}_1 C_{n-1}^{(n)}}{n(n+2)} - \frac{\hat{\tau}_2 C_{n-1}^{(n-1)}}{n(n+1)}\right] x^{n+1} + \cdots \quad (3.27)$$

Equating the corresponding coefficient of $x^{n+3}, x^{n+2} and x^{n+1}$ in (3.27) we have

$\hat{\varphi}_n = -\frac{\kappa_1^2 \tau_2}{n(n+1)R_3}$ (3.28)

where R_3 is obtained recursively as

$$R_3 = \lambda_3 - \frac{C_{n+1}^{(n+3)} R_1}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)} R_2}{C_{n+3}^{(n+3)}}$$

$$R_3 = \lambda_2 - \frac{C_{n+2}^{(n+3)} R_1}{C_{n+3}^{(n+3)}}$$

$$R_1 = \lambda_1$$

(3.29)

and

$$\lambda_1 = \left[\frac{P_{00} + (n+1)P_{11} + (n+1)P_{22}}{(n+2)(n+3)}\right]k_1$$

$$\lambda_2 = \left[\frac{P_{10} + nP_{21}}{n+2}\right]k_1 + \left[\frac{P_{00} + nP_{11} + n(n-1)P_{22}}{(n+1)(n+3)}\right]k_2$$

$$\lambda_3 = P_{20}k_1 + \left[\frac{P_{10} + (n-1)P_{21}}{n+1}\right]k_2 + \left[\frac{P_{00} + (n-1)P_{11} + (n-1)(n-2)P_{22}}{n(n+1)}\right]k_3$$

$$k_1 = C_{n-1}^{(n-1)}, k_2 = C_{n-2}^{(n-1)}, k_3 = C_{n-3}^{(n-1)}, k_4 = C_{n-4}^{(n-1)} e.\text{tc}. \tag{3.30}$$

<u>3.5</u> The case m=2, s=1

Continuing the process, using m=2 and varying s =1 in (3.24) we have obtained,

$$\hat{\varphi}_n = \left[\frac{C_{n+1}^{(n+3)}C_{n+1}^{(n+1)}}{(n+2)(n+3)C_{n+3}^{(n+3)}} - \frac{C_{n-1}^{(n+1)}}{n(n+1)}\right]\frac{k_1\tau_1}{R_4} - \frac{k_1^2\tau_1}{n(n+1)R_4} \tag{3.31}$$

With the following recursive form :

$$R_4 = \lambda_4 - \frac{C_{n+1}^{n+4}R_1}{C_{n+1}^{(n+4)}} - \frac{C_{n+1}^{n+3}R_2}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{n+2}R_3}{C_{n+2}^{(n+2)}}$$

$$R_3 = \lambda_3 - \frac{C_{n+2}^{n+4}R_1}{C_{n+4}^{(n+4)}} - \frac{C_{n+2}^{n+3}R_2}{C_{n+3}^{(n+3)}}$$

$$R_3 = \lambda_2 - \frac{C_{n+3}^{n+4}R_1}{C_{n+1}^{(n+4)}}$$

$$R_1 = \lambda_1 \tag{3.32}$$

where,

$$\lambda_1 = \left[\frac{P_{01} + (n+1)P_{12} + n(n+1)P_{22}}{(n+3)(n+4)}\right]k_1$$

$$\lambda_2 = \left[\frac{P_{00} + (n+1)P_{11} + n(n+1)P_{22}}{(n+2)(n+3)}\right]k_1 + \left[\frac{P_{01} + (n+1)P_{12} + n(n+1)P_{22}}{(n+2)(n+3)}\right]k_2$$

$$\lambda_3 = \left[\frac{P_{10} + nP_{21}}{n+2}\right]k_1 + \left[\frac{P_{00} + nP_{11} + n(n-1)P_{22}}{(n+1)(n+2)}\right]k_2 + \left[\frac{P_{01} + (n-1)P_{12} + (n-1)(n+1)P_{23}}{(n+2)(n+3)}\right]k_3$$

$$\lambda_4 = P_{20}k_1 + \left[\frac{P_{10} + (n-1)P_{21}}{(n+1)}\right]k_2 + \left[\frac{P_{00} + (n-1)P_{11} + (n-1)(n-2)P_{22}}{(n+1)(n+2)}\right]k_3 + \left[\frac{P_{01} + (n-2)P_{12} + (n-2)(n-3)P_{23}}{n(n+1)}\right]k_4$$

$$k_1 = C_{n-1}^{(n-1)}, k_2 = C_{n-2}^{(n-1)}, k_3 = C_{n-3}^{(n-1)}, k_4 = C_{n-4}^{(n-1)}\ etc \tag{3.33}$$

3.6 The case m = 3,s = 0

From (1.2) we have for m=3 and s=0, the problem

$$Ly(x) \equiv (P_{30} + P_{31}x + P_{32}x^2 + P_{33}x^3)y'''(x) + (P_{20} + P_{21}x + P_{22}x^2)y''(x) + (P_{10} + P_{11}x)y'(x) + P_{00}y(x) = \sum_{r=0}^{F} f_r x^r$$

$$a \leq x \leq b \qquad (3.34)$$

$$y(a) = \alpha_0, y'(a) = \alpha_1, y''(a) = \alpha_2$$

With the perturbed error problem

$$I_L(e_n(x))_{n+1} \equiv \int_0^x \int_0^u \int_0^w \Big[(p_{30} + p_{31}t + p_{32}t^2 + p_{33}t^3)\big(e_n'''(t)\big)_{n+1} + (p_{20} + p_{21}t + p_{22}t^2)\big(e_n''(t)\big)_{n+1} + (p_{10} + p_{11}t)\big(e_n'(t)\big)_{n+1} + (p_{00})\big(e_n(t)\big)_{n+1}\Big] dtdwdu = -\int_0^x \int_0^u \int_0^w \Big[\tau_1 \sum_{r=0}^{n} C_r^{(n)} t^r + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)} t^r + \tau_3 \sum_{r=0}^{n-2} C_r^{(n-2)} t^r\Big] dtdwdu + \hat{\tau}_1 \sum_{r=0}^{n+4} C_r^{(n+4)} x^r + \hat{\tau}_2 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r + \hat{\tau}_3 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r$$

(3.35)

$$\big(e_n(x)\big)_{n+1} = \frac{\varphi_n x^3 \tau_{n-2}(x)}{C_{n-2}^{(n-2)}} = \frac{\varphi_n}{C_{n-2}^{(n-2)}} \sum_{r=0}^{n-2} C_r^{(n-2)} x^r$$

which yields,

$$\varphi_n = \left[\frac{C_{n+1}^{(n+3)} C_n^{(n)}}{(n+1)(n+2)(n+3)C_{n+3}^{(n+3)}} - \frac{C_{n-2}^{(n)}}{(n-1)n(n+1)}\right] \frac{k_1 \tau_1}{R_4} - \frac{k_1^2 \tau_3}{(n-1)n(n+1)R_4}$$

where the value of R is obtained recursively as in (3.32) and

$$\lambda_1 = \left[\frac{P_{00} + (n+1)P_{11} + n(n+1)P_{22} + (n-1)n(n+1)P_{22}}{(n+2)(n+3)(n+4)}\right]$$

$$\lambda_2 = \left[\frac{P_{10} + nP_{21} + n(n-1)P_{32}}{(n+2)(n+3)}\right] k_1 + \left[\frac{P_{00} + nP_{11} + n(n-1)P_{22} + n(n-1)(n-2)P_{33}}{(n+1)(n+2)(n+3)}\right] k_2$$

$$\lambda_3 = \left[\frac{P_{20} + (n-1)P_{21}}{(n+2)}\right] k_1 + \left[\frac{P_{10} + (n-1)P_{21} + (n-1)(n-2)P_{32}}{(n+1)(n+2)}\right] k_2 + \left[\frac{P_{00} + (n-1)P_{11} + (n-1)(n-2)P_{22} + (n-1)(n-2)(n-3)P_{33}}{n(n+1)(n+2)}\right] k_3$$

$$\lambda_3 = P_{30} k_1 + \left[\frac{P_{20} + (n-1)P_{21}}{(n+1)}\right] k_2 + \left[\frac{P_{10} + (n-2)P_{21} + (n-2)(n-3)P_{32}}{n(n+1)}\right] k_3 + \left[\frac{P_{00} + (n-2)P_{11} + (n-2)(n-3)P_{22} + (n-2)(n-3)(n-4)P_{33}}{(n-1)n(n+1)}\right] k_4 \qquad (3.36)$$

where,

$k_1 = C_{n-2}^{(n-2)}$, $k_2 = C_{n-3}^{(n-2)}$, $k_3 = C_{n-4}^{(n-2)}$, $k_4 = C_{n-5}^{(n-2)}$ etc

We noticed that the expression for φ_n was the same for the groupings:-

i. M = 1, S = 1, M = 2, S = 0 , (M+S=2)
ii. M = 1, S =2, M = 2, S=1, M= 3, S = 0 ,(M+S=3)

and consequently, the general expressions for φ_n corresponding to the above groupings were obtained:-

$$\varphi_n = \frac{-k_1^2 \tau_{m+s}}{\prod_{r=1}^{m}(n+s+r)\, R_{m+s+1}} \; \forall\, m+s = 1,$$

$$\varphi_n = \frac{-k_1^2 \tau_{m+s}}{\prod_{r=1}^{m}(n+s+r-1)\, R_{m+s+1}} \; \forall\, m+s = 2,$$

and

$$\varphi_n = \left[\frac{C_{n+m+s-2}^{(n+m+s)} C_{n+s}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r) C_{n+m+s}^{(n+m+s)}}\right] \frac{k_1 \tau_1}{R_{m+s+1}} - \frac{-k_1^2 \tau_{m+s}}{\prod_{r=1}^{m}(n+s+r-2) R_{m+s+1}} \quad \forall\, m+s = 3 \qquad (3.37)$$

where $k_1 = C_{n-m+1}^{(n-m+1)}$

Thus, from (2.5) we have the following expression for ε^* :-

$$\varepsilon^* = \frac{-k_1^2 \tau_{m+s}}{\prod_{r=1}^{m}(n+s+r)\, R_{m+s+1}} \forall\, m+s=1,$$

$$\varepsilon^* = \frac{-k_1^2 \tau_{m+s}}{\prod_{r=1}^{m}(n+s+r-1)\, R_{m+s+1}} \qquad \forall\, m+s=2,$$

$$\varepsilon^* = \left[\frac{C_{n+m+s-2}^{(n+m+s)}\, C_{n+s}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r) C_{n+m+s}^{(n+m+s)}}\right] \frac{k_1 \tau_1}{R_{m+s+1}} - \frac{-k_1^2 \tau_{m+s}}{\prod_{r=1}^{m}(n+s+r-2) R_{m+s+1}} \quad \forall\, m+s=3 \tag{3.38}$$

IV. Numerical Examples

We consider here some selected examples with our results of obtained in the preceding sections for m + s ≤ 3. The exact errors are obtained as

$\varepsilon^* = \max_{0\le x\le 1}\{|y(x_k) - y_n(x_k)|\}, \quad 0 \le x \le 1,\ \{x_k\} = \{0.01k\},$ for k = 0(1)(100)

Example 5.1

$$Ly(x) = y^1(x) - x^2 y(x) = 0$$

$$y(0) = 1$$

with exact solution $y(x) = \exp\left(\frac{1}{3}x^3\right), \quad 0 \le x \le 1.$

Here, m = 1, s = 2

The numerical example is presented in table 5.1 below.

Example 5.2

A Second Order Homogeneous Constant Coefficient problem

$$Ly(x) = y''(x) - y(x) = 0$$

$$y(0) = 1, \quad y'(0) = 1$$

With analytical solution $y(x) = e^x, 0 \le x \le 1.$ For this case m = 2 and s = 0. See Table 5.2 below for numerical results.

Example 5.3

Third Order Non-Homogenous Constant Coefficient Problem.

$$Ly(x) = y'''(x) - 8y'(x) = -6x^2 + 9x + 2$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = \frac{11}{128}$$

With exact solution $y(x) = \frac{11x^2}{256} + \frac{7x^2}{32} - \frac{x^4}{16}$ $0 \le x \le 1$. for this problem m = 3 and s = 0. See Table 5.3 below for numerical examples.

Table 5.1

Error and Error Estimates for Example 5.1

N \ Error	6	7	8	9
Approximate Error	1.46×10^{-6}	2.18×10^{-7}	1.48×10^{-10}	4.13×10^{-12}
Exact Error	2.40×10^{-6}	5.23×10^{-7}	4.60×10^{-10}	2.34×10^{-11}

Table 5.2

Error and Error Estimates for Example 5.2

N \ Error	7	8	9	10
Approximate Error	2.26×10^{-11}	4.87×10^{-12}	1.09×10^{-14}	2.06×10^{-15}
Exact Error	3.61×10^{-9}	1.62×10^{-11}	4.26×10^{-13}	4.24×10^{-15}

Table 5.3

Error and Error Estimates for Example 5.3

N \ Error	2	3	4	5
Approximate Error	9.08×10^{-4}	6.03×10^{-5}	4.49×10^{-6}	2.84×10^{-7}

Exact Error	7.10×10^{-5}	3.31×10^{-5}	4.16×10^{-6}	8.00×10^{-8}

V. Conclusion

The integrated formulation of the tau method of initial value problems (IVPs) for the Ordinary differential equations characterized by a maximum of two overdetermination number such that $m + s \leq 3$ (m, order s, number of overdetermination) has been presented. The method closely estimated the error involved in the approximants. This is obvious from the numerical evidencies obtained from some selected problems.

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