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The Method Of Walther Heinrich Wilhelm Ritz Will Be Used To Find The Solution Of The Heat Equation That Considers The Internal Generation

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Abstract:
The Jean-Baptiste Joseph Fourier and Laplace methods are powerful tools for solving differential equations, but they have their limits. When the boundary conditions of a problem are complex, variational methods such as Ritz's method offer an alternative route.
To better understand how Ritz's method works, let's consider a simple problem: the temperature distribution in a rod with an internal heat source. By solving this problem both analytically and in the approximate Ritz method, we can compare the results and evaluate the accuracy of the latter. This comparison will allow us to determine whether Ritz's method is a reliable tool for solving more complex problems.
***Background**: Walther Ritz developed a systematic approach to solving variational problems. His methods revolutionized variational calculus, transforming it from a theoretical tool to one of great practical use. The importance of his contributions lies in the ability to approximate solutions of ordinary or partial second-order differential equations, thus offering approximate analytical solutions to complex problems.*
***Materials and Methods**: The objective of this study is to evaluate two test functions using the Ritz method, comparing them graphically with the analytical solution corresponding to the problem.*
***Results**: Two test functions were applied, one polynomial and one trigonometric, both satisfying the boundary conditions with a single term. These functions were used to approximate the solution*
***Conclusion:** Graphically, both test functions approximated the analytical solution, showing very small error.*
***Key Word**: Walther Heinrich Wilhelm Ritz; Jean-Baptiste Joseph Fourier; Pierre-Simon Laplace; Analytical solution; Approximation.*

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I. Introduction

A Hilbert space, according to David Hilbert's definition, is a complete metric space (X,d) equipped with an inner product. This structure induces a norm on the space, as described[1].

$$L^2(a,b) := \left\{ f :]a,b[\longrightarrow \mathbb{C} / \int_a^b |f(x)|^2 < \infty \right\} \quad (1)$$

It can be shown[1,2,3] that $L^2(a,b)$ is a Hilbert space when it has an inner product. Walter Ritz was the first to think of defining i by minimizing a special function. Later, Boris Galerkin did something much more general that worked for any kind of equation[4].

II. Material And Methods

Steady-state heat conduction, a phenomenon of great interest in engineering, can be effectively addressed by variational formulations. The Ritz method, a functional minimization technique proposed by Walter Ritz, has established itself as a fundamental tool in the numerical resolution of these problems. The Dirichlet problem, a particular case of this type, has been the subject of in-depth study. Despite the apparent intuition about the existence and uniqueness of its solution, Weierstrass identified a flaw in Dirichlet's original proof. Hilbert subsequently managed to establish a rigorous proof, showing that the existence of the solution is conditioned by the regularity of the contour and the boundary conditions[5,6,7,8].

Ritz proposed a simple and efficient method to approximately solve the variational form of the steady-state heat conduction equation[9,10]. In this work, this method will be used to address heat conduction problems with internal generation in steady state in finite regions.

$\nabla^2 T(r) + A(r)T(r) + \frac{1}{k} g(r) = 0$ in the region $\mathbb{R}$ (2)

$\frac{\partial T}{\partial n_i} + H_i T = f_i(r_s)$ on the bondary S_i (3)

where i is an index ranging from 1 $to\ s$, where s represents the number of continuous boundary surfaces in the región $\mathbb{R}$, $\frac{\partial}{\partial n_i}$ The normal derivative is the derivative along the normal drawn outward from the surface of the region. The equivalent variational form has been determined for case (2), so the corresponding one for this case is:

$$I = \int_R \left[(\nabla T)^2 - AT^2 - \frac{2}{k} gT \right] dv + \sum_{i=1}^{s} \int_{s_i} \left(H_i T^2 - 2 f_i T \right) ds_i \qquad (4)$$

Where:

$\int_s \equiv \sum_{i=1}^{s} \int_{s_i}$ (5)

The variational expression, which incorporates the boundary conditions of the problem, presents difficulties in obtaining an exact analytical solution. Therefore, the Ritz method is used as an approximation tool. This method begins with the choice of a test function that, in addition to containing adjustable parameters, must meet the boundary conditions (3). It is important to note that the differential equation (2) is not an indispensable requirement for this approach. Where the function Ψ_0 meets the non-homogeneous part of the boundary conditions equation (3).

$$\frac{\partial \Psi_0}{\partial n_i} + H_i \Psi_0 = f_i \qquad (6)$$

Functions are selected φ_j $j = 1,2,..,n$, linearly independent and known, so that they satisfy the homogeneous part of the boundary conditions equation (3).

$$\frac{\partial \phi_j}{\partial n_i} + H_i \phi_j = 0 \qquad (7)$$

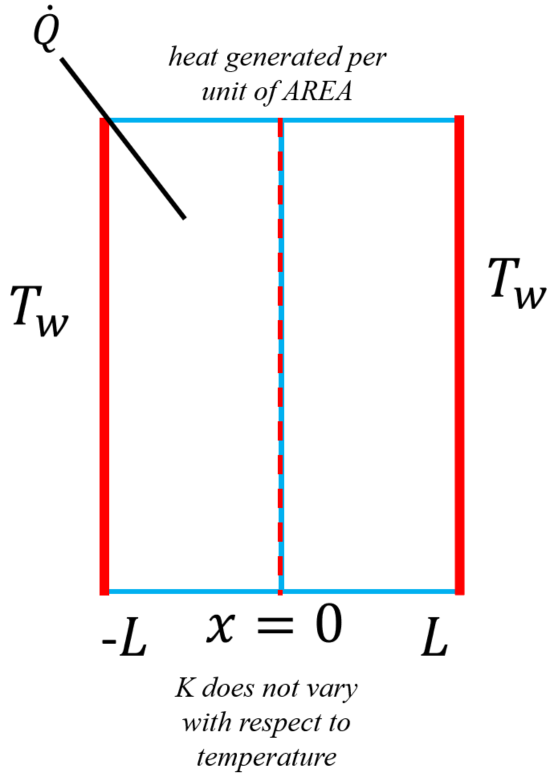
Then, the test solution for equation (4) satisfies the boundary conditions equation (3) for any choice of the coefficients s_i, $i = 1,2,..$,. If the boundary conditions of the problem are homogeneous, the function H_i is superfluous and it is only necessary to determine the functions φ_j $j = 1,2,..,n$. It is required that the functions φ_j $j = 1,2,..,n$ have continuous ordinary and partial derivatives up to the second order with respect to the spatial variables.

Once the test solution has stabilized $\tilde{T}_n(r)$, Ritz's method for determining coefficients c_j consiste en sustituir $\tilde{T}_n(r)$ the variational expression equation (4), which requires the minimization of a functional with respect to these coefficients. This process leads to a system of algebraic equations that, when solved, provides the optimal values of the coefficients c_j $j = 1..,n$.

This procedure results in a system of n algebraic equations that allow the calculation of the n unknown coefficients, offering an approximate solution to the variational problem. This approximation is justified by the fact that the solution found is a stationary point of the functional $I(c_1, c_2, .., c_n)$ restricted to the space of test functions. For an evaluation of the error in this method, see[6].

Selecting the function family ϕ_j constitutes a fundamental aspect in this method. These functions are required to form a complete set in the region of interest, i.e., any sufficiently smooth function in that region can be approximated arbitrarily well by finite linear combinations of the functions. ϕ_j $j = 1, 2, \ldots n$. The choice of these functions, which may be polynomials, trigonometric functions, or special functions, will depend on the nature of the physical problem. It is important to note that an appropriate selection of these functions is essential to obtain a good approximation to the solution. As an illustration, consider the case of a plane wall with a uniformly distributed heat source, with a thickness of $2L$ in the direction x, as shown in

Figure no. 1 Graphical representation with internal heat generation.



The partial differential equation that describes this phenomenon, considering the internal heat generation, is the following:

$$\frac{d^2T}{dx^2} + \frac{Q}{K} = 0 \qquad (8)$$

Boundary conditions

$$T = T_w \text{ en } x = \pm L \qquad (9)$$

The general analytical solution of the differential equation (8) is given by the following expression:

$$T(x) = -\frac{Q}{2K}x^2 + c_1x + c_2 \qquad (10)$$

Solution using the boundary conditions (9) we obtain:

$$T(x) = T_w + \frac{1}{2}(L^2 - x^2) \qquad (11)$$

with the boundary conditions given by equation (9). the variational formulation for equation (12) has the form:

$$I = \int_{-L}^{L}\left[(\nabla T)^2 - \frac{2}{k}QT\right]dx \qquad (12)$$

The Ritz method is being used to find a solution to the proposed differential equation.

$$\nabla^2T(r) + A(r)T(r) + \frac{1}{k}g(r) = 0 \text{ in } -L \leq x \leq L \qquad (13)$$

taking $r = x$, $A(r) = 1$, $\mathrm{T(r)} = 0$, $g(r) = Q$ you have

$$\nabla^2T(x) + \frac{1}{k}Q = 0 \text{ in } -L \leq x \leq L \qquad (14)$$

Given the boundary conditions given by equation (9), the variational formulation for equation (14) takes the following formulation:

$$I = \int_{-L}^{L}\left[(\nabla T)^2 - \frac{2}{k}QT\right]dx \qquad (15)$$

Given the boundary conditions given by equation (9), the variational formulation for equation (15) takes the following formulation:

$$\frac{\partial I}{\partial c_i} = 0 \qquad (16)$$

For the case under study, two test functions are proposed

$$\varphi_1(x) = (L^2 - x^2) \qquad (17)$$

$$\varphi_2(x) = cos\left(\frac{\pi x}{2L}\right) \qquad (18)$$

These functions also satisfy the boundary conditions given by equation (9)

$\tilde{T}_1 = T_w + C_1\varphi_1(x)$ (19)

$\tilde{T}_2 = T_w + C_2(x)\varphi_2(x)$ (20)

These solutions also satisfy the boundary conditions given by equation (19)

$\tilde{T}_1 = T_w + \frac{Q}{2K}(L^2 - x^2)$ (21)

By carrying out an operation analogous to that performed for equation (20), we have

$\tilde{T}_2 = T_w + \frac{16QL^2}{K\pi^3}cos\left(\frac{\pi x}{2L}\right)$ (22)

To graphically observe the behavior of the analytical solution with the two approximate solutions, the following values are taken: $Q = 1, K = 1$, $-1 \leq x \leq 1$, L = 1, $T_w = 0$

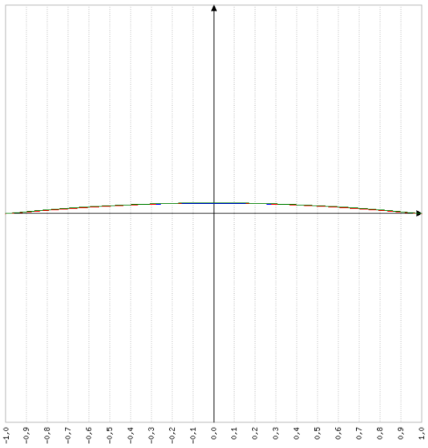
$\tilde{T}_1(x) = \frac{1}{2}(1 - x^2)$ (23)

$\tilde{T}_2(x) = \frac{16}{\pi^3}cos\left(\frac{\pi x}{2L}\right)$ (24)

$T(x) = \frac{1}{2}(1 - x^2)$ (25)

A graphical representation of the analytical solution of equation (21) is presented in figure (1), together with the approximate solutions calculated using equation (22-24).

Figure no 1. Blue color analytical solution, $\tilde{T}_1(x)$ red color and $\tilde{T}_2(x)$ green



III. Result

In order to apply the Ritz method, it is essential to have a variational formulation of the problem to be studied. As test functions, polynomials or trigonometric functions can be selected, provided that they satisfy the imposed boundary conditions. By introducing these functions into the variational formulation and calculating the corresponding coefficients, the application of the method is completed. Figure no. 1 presents the analytical solution together with two test functions, and it is possible to increase the number of terms in the latter. It is clearly observed how the approximations converge towards the exact solution

IV. Discussion

In order to apply the Ritz method, it is essential to have a variational formulation of the problem to be studied. As test functions, polynomials or trigonometric functions can be selected, provided that they satisfy the imposed boundary conditions. By introducing these functions into the variational formulation and calculating the corresponding coefficients, the application of the method is completed. Figure no. 1 presents the analytical solution together with two test functions, and it is possible to increase the number of terms in the latter. It is clearly observed how the approximations converge towards the exact solution

V. Conclusion

This method is very effective for obtaining approximate solutions to problems involving second-order ordinary differential equations, as well as partial differential equations such as Laplace and Poisson. One of its main advantages compared to numerical methods is that it allows the solution to be found in the entire domain of study. Numerical methods, on the other hand, discretize the domain, which may cause the exact value sought to not coincide with a node of the discretization.

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